# ESSAYS

ON SEVERAL Barker

Curious and Useful SUBJECTS,

In SPECULATIVE and MIX'D

# MATHEMATICKS.



Illustrated by a Variety of EXAMPLES.

By *THOMAS SIMPSON.*



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M.DCC.XL.

TO

# *FRANCIS BLAKE*,

OF

*Twisel*, in the County of *Durham*, Esq;

SIR,

AS our private Correspondence occasioned my Drawing up several of the following Papers, I thence claim a Sort of a Right to address them to You: But I well know the

# PREFACE.

THE Reader, I presume, will excuse me, if, instead of acquainting him, in the usual Way, with the many weighty Reasons that induced me to publish the following Sheets, I shall take up no more of his Time than to give a concise Account of the Nature and Usefulness of the several Papers that compose this Miscellany, in the Order they are printed.

The first, then, is concerned in determining the Apparent Place of the Stars arising from the progressive Motion of Light, and of the Earth in its Orbit; which, though it be a Matter of great Importance in Astronomy, and allowed one of the finest Discoveries, yet had it not been fully and demonstratively treated of by any Author, or indeed thrown into any Method of Practice. Now, however, I must not omit to acknowledge, that in the last Volume of the Memoirs of the ROYAL ACADEMY of SCIENCES, for the Year 1737.

*lately published at* Paris, *and brought hither a few Weeks since, there is a Paper on this Subject by Monsieur* Clairaut, *a very eminent Mathematician of that* Academy; *to which he subjoins a Set of Practical Rules for the* Aberration *in* Right-Ascension *and* Declination *only; wherein most of his Analogies are exactly the same as those inserted in this Book, with which Dr.* Bevis *favoured me: For which Reason I think it proper to assure my Readers, that my Paper, together with the Doctor's Rules, were quite printed off, and in the Hands of several Friends, who desired them, before* Christmas 1739. *when the Severity of the Season interrupted for a considerable Time the Impression of this Treatise.* 1

*The second Paper, treats of the Motion of Bodies affected by Projectile and Centripetal Forces; wherein the Invention of Orbits and the Motion of Apsides, with many others of the most considerable Matters in the First Book of Sir* Isaac Newton's PRINCIPIA, *are fully and clearly investigated.* 21

*The Third, shews how, from the Mean Anomaly of a Planet given, to find its true Place in its Orbit, by three several Methods; but what may best recommend this Paper, is the Practical Rule in the annexed Scholium, which will, I hope, be found of Service.* 41

*The Fourth, includes the Motion and Paths of Projectiles in resisting Mediums, in which not only the Equation of the Curve described according to any Law of Density, Resistance, &c. but all the most important Matters, upon this Head, in the Second Book of the above-named illustrious Author, are determined in a new, easy, and comprehensive Manner.* 52

*The Fifth, considers the Resistances, Velocities, and Times of Vibration, of pendulous Bodies in Mediums.* 65

*The*

*The Sixth, contains a new Method for the Solution of all Kinds of Algebraical Equations in Numbers; which, as it is more general than any hitherto given, cannot but be of considerable Use, though it perhaps may be objected, that the Method of Fluxions, whereon it is founded, being a more exalted Branch of the Mathematicks, cannot be so properly applied to what belongs to common Algebra.* 81

*The Seventh, relates to the Method of Increments; which is illustrated by some familiar and useful Examples.* 87

*The Eighth, is a short Investigation of a Theorem for finding the Sum of a Series of Quantities by Means of their Differences.* 94

*The Ninth, exhibits an easy and general Way of Investigating the Sum of a recurring Series.* 96

*These three last Papers relate chiefly to the Inventions of Others: As they are all of Importance, and are required in other Parts of the Book, I could not well leave them entirely untouch'd; and if I shall be thought to have thrown any new Light upon them, that may benefit young Proficients, I have my End.*

*The Tenth, comprehends a new and general Method for finding the Sum of any Series of Powers whose Roots are in Arithmetical Progression, which may be applied with equal Advantage to Series of other Kinds.* 98

*The Eleventh, is concerned about Angular Sections and some remarkable Properties of the Circle.* 105

*The Twelfth, includes an easy and expeditious Method of Reducing a Compound Fraction to Simple Ones; the first Hints whereof I freely acknowledge to have received from Mr.* Muller's *ingenious Treatise on Conic Sections and Fluxions.* 110

*The Thirteenth and last, containing a general Quadrature of Hyperbolical Curves, is a Problem remarkable enough, as well on account of its Difficulty, as its having exercised the Skill of several great Mathematicians; but as none of the Solutions hitherto published, tho' some of them are very elegant ones, extend farther than to particular Cases, except that given in* Phil. Transf. N°. 417. *without Demonstration, I flatter myself that this which I have now offered, may claim an Acceptance, since it is clearly investigated by two different Methods, without referring to what hath been done by Others, and the general Construction rendered abundantly more simple and fit for Practice than it there is.* 121

# ESSAYS

## *On ſeveral Curious and Uſeful Subjects in* Speculative *and* Mixt Mathematicks.

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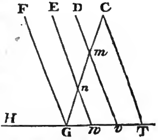
### Of the Apparent Places of the FIXED STARS, ariſing from the *Motion of Light*, and the *Motion of the Earth in its Orbit*.

#### PROPOSITION I.

*If the Velocity of the Earth in its Orbit bears any ſenſible Proportion to the Velocity of Light, every Star in the Heavens muſt appear diſtant from its true Place; and that by ſo much the more, as the Ratio of thoſe Velocities approaches nearer to that of Equality.*



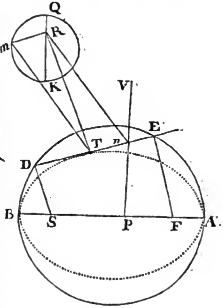

FOR, if while the Line CG is deſcribed by a Particle of Light coming from a Star in that Direction, the Eye of an Obſerver at T be carry'd, by the Earth's Motion, thro' TG; and CT be a Tube made uſe of in obſerving; and a Particle of Light, from the ſaid Star, be

juſt entering at C the End of its Axis; then when the Eye is arrived at $v$, the Tube will have acquired the Poſition $v$ D parallel to TC, and the ſaid Particle will be at the Point $m$, where the Line CG interſects the Axis of the Tube; becauſe $GT : GC :: Tv : Cm$. Let now the Tube, by the Earth's Motion, be brought into the Poſition E$w$; then becauſe $GT : GC :: Tw : Cn$, the Particle will be at $n$, and therefore is ſtill in the Axis of the Tube: Therefore when it enters the Eye at G, as it has all the Time been in the Axis of the Tube, it muſt conſequently appear to have come in the Direction thereof, or to make an Angle with TH, the Line that the Earth moves in, equal to CTH, which is different from what it really does, by the Angle GCT: Whence it is evident that, unleſs the Earth always moves in a Right Line directly to or from a given Star (which is abſurd to ſuppoſe) that Star muſt appear diſtant from its true Place; and the more ſo, as the Velocity of the Earth (in reſpect of that of Light) is increaſed. And the ſame muſt neceſſarily be the Caſe when the Obſervation is made by the naked Eye; for the Suppoſition and Uſe of a Tube neither alters the real nor apparent Place of the Star, but only helps to a more eaſy Demonſtration.

PRO-

## PROPOSITION II.

*To find the Path which a Star, thro' the aforesaid Cause, in one entire Annual Revolution of the Earth, appears to describe.*



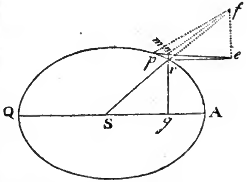

LET ATBA be the Orbit of the Earth; S the Sun in one Focus; F the other Focus; T the Earth moving in its Orbit from A towards B; DT*n* a Tangent at T; and SD, FE Perpendiculars thereto: Let Q*m*KRQ be Part of an indefinite Plane parallel to that of the Ecliptick, passing thro' R the Centre of the given Star; and take T*n* to TR, as the Velocity of the Earth in its Orbit at T, to that of a Particle of Light coming from the said Star: Let T*m* be parallel to *n*R; P*n*V perpendicular to AB; and QRK parallel to P*n*V: Then from the foregoing Proposition it is manifest, that a Ray of Light coming from R to the Earth at T, will appear as if it proceeded from *m*, where the Line T*m*, produced, intersects the said parallel Plane; and therefore, because T*m* is parallel to R*n*, and any Parallelogram, intersecting two parallel Planes, cuts them alike in every respect, it is evident that R*m* must be equal to T*n*, and QR*m* to V*n*D; wherefore since D and P are equal to

two Right Angles, DSP and $DnP$ muſt be equal, alſo, to two Right Angles, and conſequently $QRm$ ($= VnD$) $=$ DSP $=$ AFE. But $Tn$ or $Rm$, expreſſing the Celerity of the Earth at T, is known to be inverſely as SD; or, becauſe $SD \times FE$ is every where the ſame, directly as FE; wherefore the Angles AFE, $QRm$ being every where equal, and $Rm$ in a conſtant Proportion to FE, the Curve $QmK$ deſcribed by $m$, the apparent Place of the Star in the ſaid parallel Plane, will, it is manifeſt, be ſimilar in all Reſpects to AEB deſcribed by the Point E: But this Curve is known to be a Circle; therefore $QmK$ muſt likewiſe be a Circle, whoſe Diameter QRK is divided by R, the true Place of the Star, in the ſame Proportion as the Tranſverſe Axis of the Earth's Orbit is divided by either of its Foci. Wherefore, foraſmuch as a ſmall Part of the circumjacent Heavens may, in this Caſe, be conſidered as a Plane paſſing perpendicular to a Line joining the Eye and Star, it follows from the Principles of Orthographic Projection, that the Star will be ſeen in the Heavens as deſcribing an Ellipſis, whoſe Center (as the Excentricity of the Orbit is but ſmall) nearly coincides with the true Place of the Star, except the ſaid Place be in the Pole or Plane of the Ecliptick; in the former of which Caſes the Star will appear to deſcribe a Circle, and in the latter an Arch of a great Circle of the Sphere, which by Reaſon of its Smallneſs may be conſidered as a Right Line. But theſe Concluſions will perhaps appear more plain from the next Propoſition, where for the Sake of Eaſe and Brevity, the Earth is conſidered as moving in an Orbit perfectly circular, from which her real Orbit does not greatly differ.

PRO-

## PROPOSITION III.

*Having given, from Experiment, the Ratio of the Velocity of Light to that of the Earth in its Orbit, and the true Places of the Sun and a Star; to find the apparent Place of the Star from thence arising.*

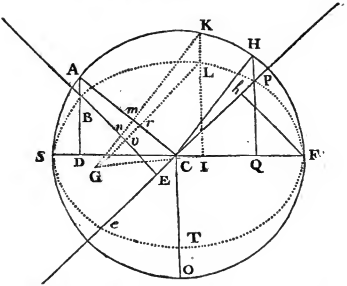



LET A $r$ Q A be the Earth's Orbit, confidered as a Circle; S the Sun in the Center thereof; $r$ the Earth moving about the fame from A towards Q; $re$ a Line, which being produced, fhall pafs thro' the Ecliptick Place of the given Star; A S parallel, and $qr$ perpendicular, thereto: Let $ef$ be perpendicular to the Plane of the Ecliptick, fo that $rf$ being equal to S$r$ or Radius, $re$ may be the Cofine of the Latitude of the given Star: This being premifed, it is manifeft that the true Place of the Star, from the Earth, will be in the Direction $rf$, and with Refpect to the Ecliptick, in the Line $re$; therefore the Angle S$re$ ($=$ QS$r$) being the Difference of Longitudes of the Sun and Star, is given by the Queftion. Let $rg$, the Sine of the Supplement of this Angle, be denoted by $b$; its Cofine S$g$, by $c$; the Sine of the given Latitude, or $fe$, by $s$; and the Radius S$r$, or $fr$, by Unity; and while a Particle of Light is moving along $fr$, let the Earth be fuppofed to be carry'd in

its Orbit from $r$ to $p$, over a Diſtance ſignified by $r$; and, $pe$, $pf$ being drawn, make $rn$ and $nm$ perpendicular thereto: Then becauſe of the exceeding Smallneſs of $pr$ it may be conſidered as a Right-Line; and we ſhall have $1\ (Sr) : b\ (rg) :: r\ (pr) : rb\ (= pn)$; and $1 : r :: c : rc\ (= rn)$ (by the Similarity of the Triangles $prn$, $Srg$); whence as $1\ (fp,)$ to $s\ (fe)$ ſo is $rb$, to $rbs = (nm)$ the Sine of the Angle $nfm$: But ſince the Sine or Tangent of a very ſmall Arch differs inſenſibly from the Arch itſelf, theſe Values $rc$ and $rbs$ may be taken as the Meaſures of the Angles $rfn$, and $nfm$: Hence we have, as the Semi-Periphery $ArQ$ ($= 3.14159$, &c.) to 648000 (the Seconds in 180 Degrees,) ſo is $rc$ to $\frac{648000\ rc}{3.14159, \&c.}$ (the Number of Seconds in the Angle $rfn$;) and as 3.14159 &c.: $648000 :: rsb : \frac{648000\ rsb}{3.14159, \&c.} = nfm$: Therefore, as the Earth moves from $r$ to $p$ while a Particle of Light is deſcribing $fr$, it is manifeſt from the aforeſaid Propoſition, that the Star will appear removed from the great Circle paſſing through its true Place, and the Pole of the Ecliptick by $\frac{648000\ rc}{3.14159, \&c.}$ Seconds; and to have its Latitude increaſed by $\frac{648000\ rsb}{3.14159, \&c.}$ Seconds. *Q. E. I.*

COROL.

## COROL. I.




HENCE if C be the true Place of the Star, S C F its Parallel of Latitude; and about C, as a Centre, the Ellipsis F P S T F, and Circle F H S O F be described so, that F C may be $=\frac{648000\,r}{3.14159,\&c.}$ and T C, the Semi-Conjugate Axis, in proportion thereto, as $s$ to $1$; and if the Angle S C H be taken equal to the Difference of Longitudes of the Sun and Star; then in the Point P, where the elliptical Periphery is intersected by the Right Line H Q falling perpendicularly on FS, the Star will appear to be posited. For as 1 (Radius) : $b$ (Sine of Q C H) :: C H : $b \times$ C H = H Q; but by the Relation of the two Curves, C H : C T :: $b \times$C H (=H Q) : P Q, that is, by Construction, $1 : s :: \frac{648000\,r}{3.14159,\&c.} \times b : \frac{648000\,r\,s\,b}{3.14159,\&c.}$ = P Q;

$= PQ$; again as 1 (Radius) : $c$ (the Cosine of $QCH$) :: $\frac{648000\,r}{3.14159,\&c.}$ $(= CH)$ : $\frac{648000\,rc}{3.14159,\&c.} = CQ$; which Expressions are the very same as those above determined.

## COROL. II.

THEREFORE it follows, that while the Sun appears to pursue his Course thro' the Ecliptick, the Star will be seen as moving from F towards L and S, and so on, 'till it hath described the whole elliptical Periphery FLSTF; that its Latitude will be the the least at T; and its apparent Longitude the greatest possible, when the Angle SCH, shewing the Distance of the Sun and Star in the Ecliptick, is equal to two right ones. It also follows, that the greater Axis of the Ellipses, which all Stars appear to describe, are equal, and found by Observation to amount to 40 ½ Seconds of a great Circle, very nearly; the Term 20″, 25 which frequently occurs in the practical Rules hereto annext, being put for the half thereof. It follows moreover, that the greatest Aberrations, or *Maxima*, in Longitude, will be as the Cosines of the Latitudes inversely; and the *Maxima* in Latitude, as the Sines of the same Latitudes directly.

## COROL. III.

HENCE may also be found the Stars apparent Right Ascension and Declination; for let ÆCP be the Parallel of the Stars Declination, P the apparent Place of the Star when in that Parallel; make CA perpendicular to CH, ABD to SF, and BE to PC; and let HK, or the Angle HCK be any Distance gone over by the Earth in the Ecliptick, while the Star by its apparent Motion moves thro' the corresponding Distance PL: Let $KmnG$ be parallel to HC,

and

and $Lrv$ to PC: Then, forasmuch as KL is parallel to HP, the Triangles GKL, CHP must be equiangular, and therefore GL : CP :: KL : HP; but KL is to HP, as LI to QP, by the Property of the Curve; whence it will be GL : CP :: LI : QP: Wherefore, the Sides GL, IL, CP, QP about the equal Angles GLI, CPQ being proportional, the Triangles GLI, CPQ must be similar, and therefore the Angle GIL a right one, and consequently the Right Line SF the Locus of the Point G. Therefore, as the Angles $n$, $m$, $r$, $v$ are all given, or continue invariable, let the Angle SCK, or the ecliptick Distance of the Sun and Star be what it will, the Ratio of $Cm$ to CG will always be given; but the Ratio of CG to $Cr$ is given; therefore the Ratio of $Cm$ to $Cr$ is likewise given: Hence, because $rv$ is parallel to CE, the Ratio of $Cm$ to $Ev$ will be given. But $Ev$ is the Difference of the true and apparent Declinations; and $Cm$, as the Sine of the Angle HCK: Whence it is manifest, that the Aberration of Declination, at any Time, is as the Sine of the Sun's Elongation from either of the two Points wherein he is, when the true and apparent Declinations are the same; and therefore $Cm$ will be to $Ev$, or AC to EB, the greatest Aberration, as QH to $Fb$, that is, as the Sine of HCF to the Sine of PCQ: But PCQ, being equal to the Angle of Position, is given, whose Tangent, it is obvious, is to the Tangent of HCF, as QP to QH, or as CT to CO, or lastly, (by Construction) as the Sine of the Star's Latitude to Radius: Hence the Angle HCF is given, from which, by Help of the foregoing Theorem or Proportion, the required Aberration of Declination at any Time, and in any Case, may be readily obtained.

In like manner other Proportions may be derived for finding the Aberration of Right Ascension; it being easy to prove that it will be as the Sine of the Sun's Elongation from where

he is, when the true and apparent right Afcenfions are the fame; but the Method of Demonftration being the fame as above, it will be needlefs to repeat it.

I fhall therefore now proceed to illuftrate the foregoing Doctrine by the practical Solutions of the feveral Problems depending thereon, as they were drawn up and communicated by Dr. *John Bevis*, with fuitable Examples of feveral Stars, which, among many others, He has carefully obferved with proper Inftruments, and thereby, the firft of any one that I know of, experimentally prov'd, that the Phœnomena are univerfally as conformable to the Hypothefis in Right Afcenfions, as the Rev. Mr. *Bradley*, to whom we owe this great Difcovery, had before found them to be in Declinations.

PRAC-

# PRACTICAL RULES

For Finding the

# ABERRATIONS

OF THE

# FIXT STARS

FROM

## The Motion of Light, and of the Earth in its Orbit,

IN

## Longitude, Latitude, Declination, and Right Afcenfion.

---

### SYMBOLS.

*A*, the Aberration at any given Time.

*M*, the greateſt Aberration, or *Maximum*.

☉, the Sun's Place in the Eclipticks when the Star's Apparent Longitude, Latitude, Declination, or Right Afcenfion, being the fame as the True, tends to Excefs.

P, the Star's Angle of Pofition.

Z, the Sun's Elongation from its neareſt Syzygy with the Star, at the Time of ☉.

*For*

## *For the Abberration in* LONGITUDE.

$\odot$ is always 3 Signs after the Star's true Place in the Ecliptick.

### PROB. I. *To find* M.

RULE.

*Cos.* Star's Latit. : *Rad.* :: 20″, 25 : M.

EXAMPLE in *γ Ursæ minoris.*

OPERATION.

| | |
|---|---|
| Log. *Cos. Ar. Com.* Star's Latit. 75° 13′ | 0.5932 |
| + Log. 20″, 25 | 1.3064 |
| = Log. M 79″, 36 | 1.8996 |

### PROB. II. *To find* A.

RULE.

*Rad.* : *Sin.* Sun's Elongat. from $\odot$ :: M : A.

EXAMPLE in the same Star.

OPERATION.

| | |
|---|---|
| Log. *Sin.* Sun's Elongat. from $\odot$ 60° 00′ | 9.9375 |
| + Log. M 79″,36 | 1.8996 |
| — Log. *Sin. Rad.* = Log. A 68″,72 | 11.8371 |

Otherwise, without M.

RULE.

*Cos.* Star's Latit. : *Sin.* Sun's Elongat. from $\odot$ :: 20″,25 : A.

Same EXAMPLE as before.

OPERATION.

| | |
|---|---|
| Log. *Cos. Ar. Com.* Star's Latit. 75° 13′ | 0.5932 |
| + Log. *Sin.* Sun's Elongat. from $\odot$ 60° 00′ | 9.9375 |
| + Log. 20″,25 | 1.3064 |
| — Log. *Sin. Rad.* = Log. A 68″,72 | 11.8371 |

*For*

## *For the* Aberration *in* LATITUDE.

☉ is always at the Sun's Oppofition to the Star.

### PROB. I. *To find* M.

RULE.

*Rad.* : *Sin.* Star's Latit. :: 20″,25 : M.

EXAMPLE γ in *Urfæ minoris.*

OPERATION.

Log. *Sin.* Star's Latit. 75° 13′ ———————— 9.9854
+ Log. 20″,25 ———————— 1.3064

— Log. *Sin.* *Rad.* = Log. M. 19″,58 ———————— 11.2918

### PROB. II. *To find* A.

RULE.

*Rad.* : *Sin.* Sun's Elongat. from ☉ :: M : A.

EXAMPLE in the fame Star.

OPERATION.

Log. *Sin.* Sun's Elongat. from ☉ 60° 00′ ———————— 9.9375
+ Log. M 19″,58 ———————— 1.2918

— Log. *Sin.* *Rad.* = Log. A 16″,96 ———————— 11,2293

Otherwife, without M.

RULE.

*Rad*$^2$. : *Sin.* Star's Latit. × *Sin.* Sun's Elongat. from ☉ :: 20″,25 : A.

Same EXAMPLE as before.

OPERATION.

Log. *Sin.* Star's Latit. 75° 13′ ———————— 9.9854
+ Log. *Sin.* Sun's Elongat. from ☉ 60° 00′ ———————— 9.9375
+ Log. 20″,25 ———————— 1.3064

— 2 Log. *Sin.* *Rad.* = Log. A 16″,96 ———————— 21,2293

E Otherwife,

Otherwise,

RULE.

*Cosec.* Star's Latit. : *Sin.* Sun's Elongat. from ☉ :: 20″,25 : A.

Same EXAMPLE as before.

OPERATION.

| | |
|---|---|
| Log. *Cosec. Ar. Com.* Star's Latit. 75° 13′ | 89.9854 |
| + Log. *Sin.* Sun's Elongat. from ☉ 66° 00′ | 9.9375 |
| + Log. 20″,25 | 1.3064 |
| = Log. A 16″96 | 1.2293 |

## *For the* Aberration *in* DECLINATION.

## PROB. I. *To find* ⊙.

RULE.

*Sin.* Star's Latit. : *Rad.* :: *Tang.* P. : *Tang.* Z.

Then, if the Star (in respect of that Pole of the Equator which is of the same Denomination as the Star's Latitude) be in a Sign.

1. Ascending, and P be acute, Z taken from the opposite to its true Place, gives ⊙,

2. Ascending, and P be obtuse, Z added to its true Place, gives ⊙,

3. Descending, and P be acute, Z added to the opposite to its true Place, gives ⊙,

4. Descending, and P be obtuse, Z taken from its true Place, gives ⊙,

provided, that its Declination and Latitude be both North, or both South: But, if one be North, and the other South, then for *its true Place*, read *opposite to its true Place*, and *vice versâ*.

EXAMPLE of *Case* I. in the Pole Star.

OPERATION.

| | |
|---|---|
| Log. *Tang.* P 75° 21′ (acute + Log. *Sin. Rad.* | 20.5827 |
| — Log. *Sin.* Star's Latit. 66° 04′ North. | 9.9609 |
| = Log. *Tang.* Z 76° 34′ | 10.6218 |

| | | | |
|---|---|---|---|
| Therefore the Star's Declin. and Latit. being both N. its Place (ascending) + 6 Signs | 8ˢ | 24° | 55′ |
| — Z | 2 | 16 | 34 |
| = ⊙. | 6 | 08 | 21 |

EXAMPLE of *Case* II. in ε *Draconis*.

OPERATION.

| | |
|---|---|
| Log. *Tang*. P 93° 50′ (obtuse) + Log. *Sin. Rad.* | 21.1739 |
| — Log. *Sin*. Star's Latit. 79° 28′ (North) | 9.9926 |
| = Log. *Tang*. Z 86° 14′ | 11.1813 |
| Therefore the Star's Decl. and Latit. being both North, its true Place (ascending) | 0ˢ 29° 12′ |
| + Z | 2 26 14 |
| = ☉ | 3 25 26 |

EXAMPLE of *Case* III. η *Ursæ majoris*.

OPERATION.

| | |
|---|---|
| Log. *Tang*. P 38° 36′ (acute) + Log. *Sin. Rad.* | 19.9022 |
| — Log. *Sin*. Star's Latit. 54° 25′ (North) | 9.9102 |
| = Log. *Tang*. Z 44° 29′ | 9.9920 |
| Therefore, the Star's Decl. and Latit. being both North, its Place (descending) + 6 Signs | 11ˢ 23° 14′ |
| + Z | 1 14 29 |
| = ☉ | 1 07 43 |

EXAMPLE of *Case* IV. in γ *Ursæ minoris*.

OPERATION.

| | |
|---|---|
| Log. *Tang*. P 94° 48′ (obtuse) + Log. *Sin. Rad.* | 21.0759 |
| — Log. *Sin*. Star's Latit. 75° 13′ (North) | 9.9854 |
| = Log. *Tang*. Z 85° 22′ | 11.0905 |
| Therefore, the Star's Decl. and Latit. being both North, its true Place (descending) | 4ˢ 17° 50′ |
| — Z | 2 25 22 |
| = ☉ | 1 22 28 |

In each of these four Examples, the Declination and Latitude are of the same Denomination; it may suffice to give one where they are of contrary Denominations.

EXAMPLE of *Case* III. in *Aldebaran*.

OPERATION.

| | |
|---|---|
| Log. *Tang*. P 9° 40′ (acute) + Log. *Sin. Rad.* | 19.2313 |
| — Log. *Sin*. Star's Latit. 5° 30′ (South) | 8.9815 |
| = Log. *Tang*. Z 60° 38′ | 10.2498 |

Therefore,

| | |
|---|---|
| Therefore, Star's Decl. being North, and its Latit. South, its Place (descending to S. Pole) | 2ˢ 06° 07′ |
| + Z | 2 00 38 |
| = ⊙ | 4 06 45′ |

## PROB. II. *To find* M.

### RULE.

*Sin.* Z : *Sin.* P :: 20″,25 : M.

EXAMPLE in γ *Ursæ minoris.*

### OPERATION.

| | |
|---|---|
| Log. *Sin.* Z *Ar. Com.* 85° 22′ | 0.0014 |
| + Log. *Sin.* P 94° 48′ | 9.9985 |
| + Log. 20″,25 | 1.3064 |
| — Log. *Sin. Rad.* = Log. M 20″,24 | 11.3063 |

## PROB. III. *To find* A.

### RULE.

*Rad.* : *Sin.* Sun's Elongat. from ⊙ :: M : A.

EXAMPLE in ε *Ursæ majoris.*

### OPERATION.

| | |
|---|---|
| Log. *Sin.* Sun's Elongat. from ⊙ 75° 31′ | 9.9860 |
| + Log. M 18″,04 | 1.2560 |
| — Log. *Sin. Rad.* = Log. A 17″,46 | 11.2420 |

Otherwise, without M.

### RULE.

*Rad.* × *Sin.* Z : *Sin.* Sun's Elongat. from ⊙ × *Sin.* P :: 20″,25 : A.

Same EXAMPLE as before.

### OPERATION.

| | |
|---|---|
| Log. *Sin.* Sun's Elongat. from ⊙ 75° 31′ | 9.9860 |
| + Log. *Sin.* P 38° 36′ | 9.9751 |
| + Log. *Sin. Ar. Com.* Z 44° 29′ | 0.1545 |
| + Log. 20″,25 | 1.3064 |
| — 2 Log. *Sin. Rad.* = Log. A 17″,46 | 21.2420 |

For

## *For the* Aberration *in* RIGHT ASCENSION.

### PROB. I. *To find* ⊙.

RULE.

*Sin.* Star's Latit. : *Rad.* : : *Cotang.* P. : *Tang.* Z.

Then, if the Star (in respect of that Pole of the Equator which is of the same Denomination as the Star's Latitude) be in a Sign

1. Ascending, and P be acute, Z added to its true Place, gives ⊙.
2. Ascending, and P be obtuse, Z taken from its true Place, gives ⊙.
3. Descending, and P be acute, Z taken from the opposite to its true Place, gives ⊙.
4. Descending, and P be obtuse, Z added to the opposite to its true Place, gives ⊙.

EXAMPLE of *Case* I. in *Sirius.*

OPERATION.

| | |
|---|---|
| Log. *Sin. Ar. Com.* Star's Latit. $39^\circ\ 32'$ (South.) ——— | 0.1962 |
| + Log. *Cotang.* P $4^\circ\ 18'$ (acute) ——— | 11.1238 |
| = Log. *Tang.* Z $87^\circ\ 16'$ ——— | 11.3200 |
| Therefore the Star's true Place (ascending) ——— | $3^s\ 10^\circ\ 29'$ |
| + Z ——— | 2 27 16 |
| = ⊙ ——— | 6 07 45 |

EXAMPLE of *Case* II. in ι *Draconis.*

OPERATION.

| | |
|---|---|
| Log. *Sin. Ar. Com.* Star's Latit. $79^\circ\ 28'$ (North) ——— | 0.0074 |
| + Log. *Cotang.* P $93^\circ\ 50'$ (obtuse) ——— | 8.8261 |
| = Log. *Tang.* Z $3^\circ\ 54'$ ——— | 8.8335 |
| Therefore the Star's true Place (ascending) ——— | $0^s\ 29^\circ\ 12'$ |
| − Z ——— | 0 03 54 |
| = ⊙ ——— | 0 25 18 |

EXAMPLE of *Case* III. $\gamma$ *Draconis.*

OPERATION.

| | |
|---|---|
| Log. *Sin. Ar. Com.* Star's Latit. 74° 28′ (North) | 0.0162 |
| + Log. *Cotang.* P 3° 36′ (acute) | 11.2012 |
| = Log. *Tang.* Z 86° 32′ | 11.2174 |
| Therefore, the Star's true Place (descending) + 6 Signs | 8ˢ 24° 23′ |
| − Z | 2 26 32 |
| = ☉ | 11 27 51 |

EXAMPLE of *Case* IV. in $\gamma$ *Ursæ minoris.*

OPERATION.

| | |
|---|---|
| Log. *Sin. Ar. Com.* Star's Latit. 75° 13′ (North) | 0.0146 |
| + Log. *Cotang.* P 94° 48′ (obtuse) | 8.9241 |
| = Log. *Tang.* Z 4° 58′ | 8.9387 |
| Therefore, the Star's true Place (descending) + 6 Signs | 10ˢ 17° 50′ |
| + Z | 0 04 58 |
| = ☉ | 10 22 48 |

## PROB. II. *To find* M.

RULE.

*Cos.* Star's Decl. × *Sin.* Z : *Cos.* P × *Rad.* :: 20″,25 : M.

EXAMPLE in the Pole Star.

OPERATION.

| | |
|---|---|
| Log. *Cos. Ar. Com.* Decl. 87° 55′ | 1.4395 |
| + Log. *Sin. Ar. Com.* Z 15° 58′ | 0.5605 |
| + Log. *Cos.* P. | 9.4030 |
| + Log. 20″,25 | 1.3064 |
| − Log. *Sin. Rad.* = Log. M. 512″,16 = 8′. 32″,16 | 12.7094 |

PROB.

## PROB. III. *To find* A.

RULE.

*Rad.* : *Sin.* Sun's Elongat. from ⊙ : : M : A.

EXAMPLE in *Lucida Aquilæ*.

OPERATION.

| | |
|---|---|
| Log. *Sin.* Sun's Elongat. from ⊙ 65° 24′ | 9.9587 |
| + Log. M 20″,18 | 1.3049 |
| — Log. *Sin. Rad.* = Log. A 18″,34 | 11.2636 |

Otherwiſe, without M.

RULE.

*Coſ.* Star's Decl. × *Sin.* Z : *Sin.* Sun's Elongat. from ⊙ × *Coſ.* P : : 20″,25 : A.

Same EXAMPLE.

OPERATION.

| | |
|---|---|
| Log. *Sin.* Sun's Elongat. from ⊙ 65° 24′ | 9.9587 |
| + Log. *Coſ. Ar. Com.* Star's Decl. 8° 12′ | 0.0045 |
| + Log. *Sin. Ar. Com.* Z 84° 36′ | 0.0019 |
| + Log. *Coſ.* P 10° 55′ | 9.9921 |
| + Log. 20″,25 | 1.3064 |
| — 2 Log. *Rad.* = Log. A 18″,34 | 21.2636 |

GENE-

## General Notes.

1. That the Rules give the Values of A and M, always in Seconds of a Degree.

2. That if the Sun's Place be in that Semicircle of the Ecliptic which precedes ☉, A muſt be taken from the Star's True Longitude, Latitude, Declination, or Right Aſcenſion, to ſhew the Apparent; but if it be in that Semicircle which follows ☉, A muſt be added.

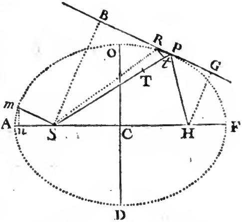
3. That ♑, ♒, ♓, ♈, ♉, ♊ are Signs Aſcending in reſpect of the North Pole, and Deſcending in reſpect of the South Pole of the Equator: And ♋, ♌, ♍, ♎, ♏, ♐, are Aſcending in reſpect of the South Pole and Deſcending in reſpect of the North Pole of the Equator.

4. That a Star may be ſo poſited, that the ſmall Ellipſe which it apparently deſcribes, may, by including, or approaching very near to the Pole of the World, make it fall under very different Conſiderations and Rules from any of the foregoing; but as the beſt Inſtruments have not diſcovered any ſuch Stars, thoſe Conſiderations and Rules have been here omitted.

## *Of the* MOTION *and* ORBITS *of Bodies affected with Projectile and Centripetal Forces.*

### PROPOSITION I.

*A Body being let go from* P, *at a given Distance* PS, *from* S, *the Center of Force, in a given Direction* P B, *with a given Velocity; To find the Conic Section it will describe, and the Periodic Time, in case it returns, the Law of Centripetal Force being as the Square of the Distance inversely, and the absolute Force given.*




LET ASF be the greater Axis of the Section, OCD the lesser, and H the upper Focus. Suppose SR indefinitely near SP, and the Area A S $m$ equal to the Area PS R; draw S B and H G perpendicular to the given Tangent B P G, and R $t$ and $m$ $n$ to S P and A F respectively: Let $r$ be the Distance that a Body would freely descend in any given Time, $m$, by an uniform Force, equal to that affecting the Projectile at any given Distance S T ($b$) from the Centre, and let $v$ be the Space that the Body would uniformly describe with the given Velocity at P, in the same Time: Call A F, $a$; O D, $e$; the Latus

G Rectum,

Rectum, or $\frac{ee}{a}$, R; S P, $d$; P R, $x$; the Periodic Time, P, and the Sine of S P B, to the Radius 1, $s$: Then it will be as $1 : s :: x : sx = Rt$; whence $\frac{sxd}{2} \left(= Rt \times \frac{SP}{2}\right)$ will expreſs either of the infinitely ſmall equal Areas PSR, $m$SA: And it will be as $v : m :: x : \frac{mx}{v}$ the Time of the Projectile's moving thro' P R, or that of its deſcribing either of the ſaid Areas by Radii drawn to the Centre of Force: Wherefore, the Diſtances which Bodies freely deſcend by uniform Forces, being as the Squares of the Times, we have, as $m^2 : r :: \frac{m^2x^2}{v^2}$ the Square of that Time, to $\frac{rx^2}{v^2}$ the Diſtance a Body would freely deſcend from the Point T in the ſame Time; but as $AS^2 : b^2\ (ST^2) :: \frac{rx^2}{v^2} : \frac{rb^2x^2}{v^2 \times AS^2} = An$, the Diſtance it would freely deſcend from the Point A in that Time; that is, in the Time the Projectile is deſcribing A$m$: Hence, becauſe $\frac{dsx}{2}$ the Area of the Triangle AS$m$, divided by $\frac{1}{2}$ AS, is $\frac{dsx}{AS} = Am$, we have $\frac{\overline{Am}|^2}{An} = \frac{d^2v^2s^2}{rbb} = R$, ſince $\frac{\overline{Am}|^2}{An}$, in the ultimate Ratio, or when A$n$ is indefinitely ſmall, will be = the Latus Rectum, let the Curve A P F be what it will. Furthermore, ſince SP + PH is $= a$, or A F, and the Angle SPB = HPG by the Property of the Curve, it will be as $1 : s :: d : sd = SB$, and as $1 : s :: a - d : s \times \overline{a - d} = HG$; but by another Property, BS × HG is = OC q or $ss \times \overline{ad - dd} = \frac{ee}{4}$; whence by ſubſtituting this Value of $\frac{ee}{4}$, in the other Equation $\frac{d^2v^2s^2}{rbb} = \frac{ee}{a}$ (= R) we get $a = \frac{d}{1 - \frac{dv^2}{4rbb}}$,

and

and therefore $e = \frac{a^{\frac{1}{2}} v s d}{b\sqrt{r}} = \left(\frac{d}{1-\frac{dvv}{4rbb}}\right)^{\frac{1}{2}} \times \frac{vsd}{b\sqrt{r}}$, and $\mathrm{PH} = \frac{d}{1-\frac{dvv}{4rbb}}$ $-d$, from which, as the Angle H P G is given, the Focus H is given likewiſe; whence the Orbit may be readily conſtructed, it being, when $\frac{d}{1-\frac{dvv}{4rbb}}$ is poſitive, an Ellipſis, when infinite, a Parabola, and when negative, an Hyperbola; wherefore, unleſs $1-\frac{dv^2}{4rbb}$ be affirmative, or $\frac{4rbb}{d}$ greater than $vv$, the Projectile can never return. Now, therefore, putting $p$ for the Area of a Circle whoſe Diameter is Unity, and ſuppoſing $\frac{4rbb}{d}$ greater than $vv$, the Area of the whole Curve, (being an Ellipſe,) will be $\frac{vsdp}{b\sqrt{r}} \times \left(\frac{d}{1-\frac{dvv}{4rbb}}\right)^{\frac{3}{2}}$ equal to $\frac{vsdpa^{\frac{3}{2}}}{b\sqrt{r}}$ $(=pae)$; but as the Area $\frac{sdx}{2}$ is to $\frac{mx}{v}$ the Time of its Deſcription, ſo is the Area of the whole Ellipſis, to $\frac{2pm}{b\sqrt{r}} \times a^{\frac{3}{2}}$ or $\frac{2pm}{b\sqrt{r}} \times \left(\frac{d}{1-\frac{dvv}{4rbb}}\right)^{\frac{3}{2}} = \mathrm{P}$, the Time of one intire Revolution. *Q. E. I.*

## COROL. I.

BECAUSE $\frac{m^2x^2}{vv}$, the Square of the Time of deſcribing the Area RSP, is to $\frac{s^2d^2x^2}{4}$, the Square of that Area, as (1) the Square of a conſtant Particle of Time to $\frac{v^2s^2d^2}{4m^2}$ the Square of the Area deſcribed in this laſt Time, it is evident that the Square laſt named will

be

be to the Latus Rectum $\frac{v^2 s^2 d^2}{rb^2}$ as $rb^2 : m^2$; which Proportion being conſtant in all Caſes relating to the ſame Center, it follows, that the principal Latera Recta of the Orbits of different Bodies, about a common Centre of Force, are directly as the Squares of the Areas deſcribed by the reſpective Bodies, in the ſame Time.

## COROL. II.

MOREOVER, ſince B S is $= sd$, and $R = \frac{d^2 v^2 s^2}{rbb}$, we have $\frac{R^{\frac{1}{2}}}{BS} = \frac{v}{b\sqrt{r}}$ and $\because$ $\frac{R^{\frac{1}{2}}}{SB}$ is to $v$, in the conſtant Ratio of 1, to $\frac{1}{b\sqrt{r}}$: Hence it appears, that the Velocities are, univerſally, in the ſubduplicate Ratio of the Parameters directly, and the Perpendiculars falling from the Center of Force on Tangents to the Places of the Bodies, inverſely, and therefore, in the ſame Orbit, the Velocity will be, barely, in the inverſe Ratio of the Perpendiculars ſo falling.

## COROL. III.

SINCE P is $= \frac{2pm}{br^{\frac{1}{2}}} \times a^{\frac{3}{2}}$, or, in a conſtant Proportion, to $a^{\frac{3}{2}}$, let $v$, $s$, and $d$, be what they will; it follows, that the Periodic Times, about the ſame Center of Force, whether in Circles or Ellipſes, will be in the ſeſquiplicate Ratio of the principal Axes.

COROL.

## COROL. IV.

BECAUSE neither the Values of $a$ nor P are affected by $s$, it follows, that the principal Axis, and the Periodic Time will be the ſame, if the Velocity at P be the ſame, let the Direction of the Projectile at that Point be what it will.

## COROL. V.

WHEN $\frac{d}{1-\frac{dvv}{4rbb}}$ $(=a)$ is $=2d$, or, which is the ſame, when $v=\sqrt{\frac{2r}{d}}$; then $d$ being the mean Diſtance or Semi-Tranſverſe, the Point P will fall in one Extreme of the Conjugate Axis, and $b\sqrt{\frac{2r}{d}}$ the Velocity there, will be juſt ſufficient to retain a Body in a Circular Orbit at that Diſtance $(d)$ from the Center of Force; and this Velocity, in reſpect of different Orbits, will, it is obvious, be inverſely as the Square Roots of the mean Diſtances: Wherefore the Velocities Bodies in Circular Orbits about a common Centre, are reciprocally in the ſubduplicate Ratio of the Radii.

## COROL. VI.

IF $v$ be $=b\sqrt{\frac{4r}{d}}$ or the Square of the Velocity be juſt twice as great as that whereby the Projectile might deſcribe a circular Orbit at its own Diſtance from the Center of Force (*Cor.*V.): then $a$, the Tranſverſe, becoming infinite, the Ellipſe degenerates into a Parabola, whoſe principal *Latus Rectum* is $4dss$; whence it appears, that the Velocity of a Body moving in a Parabola is inverſely as the Square Root of its Diſtance from the Centre of Force, and that it will be, every where, to the Ve-

locity that might carry the Projectile in a circular Orbit, at its own Diſtance from the Centre, as the Square Root of two, to one.

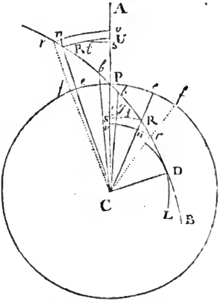
## COROL. VII.

BUT if $v$ be greater than $b\sqrt{\frac{4r}{d}}$, the Trajectory will be an Hyperbola, whoſe principal Axis is $\frac{d}{\frac{dvv}{4rbb}-1}$ $(=-a)$ as has been before intimated, and therefore $e$ $\left(=\frac{a^{\frac{1}{2}}vsd}{b\sqrt{r}}\right)$ will be $\frac{2vsd^{\frac{3}{2}}}{\sqrt{dv^2-4rbb}}$. Hence, from the Nature of the Hyperbola, if R be aſſumed for Radius, $\frac{vsd^{\frac{1}{2}}\mathrm{R}\times\sqrt{dvv-4rbb}}{2rbb}$ $\left(=\frac{e\mathrm{R}}{-a}\right)$ will be the Tangent of the Angle which the Aſymptote makes with the Axis, or the Supplement to 180° of the utmoſt Elongation the Projectile can poſſibly have from the loweſt Point of its Orbit.

PROP.

## PROPOSITION II.

*A Body is let go from* P, *at a given Distance* P C *from* C *the Centre of Force, in a given Direction* P $b$, *with a given Celerity; To find its Trajectory? the centripetal Force being as any Power* ($n$) *of the Distance, and the absolute Force at* P *given.*




LET R be a Point in the required Trajectory, and $r$ another indefinitely near it; and with the Centre C, let the Circular Arches P$ef$, RU, $vnr$, be described, and having drawn CR$e$, C$rf$, &c. let $CP = a$, CR ($= CU$) $= x$, $Pe = A$, $Rn$ ($= Uv$) $= \dot{x}$, $rn = \dot{y}$, $Rr = \dot{z}$, and $s =$ the Sine of the Angle CP$b$ to the Radius 1; and let P$b$ ($m$) be the Space that might be described in (1) a given Particle of Time with the given Celerity; and $r$ the Distance a Body would freely descend in that Time, by an uniform Force equal to that affecting the Projectile at P: Then the Space which would be uniformly described in that same Time, with the Celerity acquired by descending thro' the said Distance $r$, it is well known, will be equal to $2r$. But, from hence to

to find the Celerity at R, with the fame Velocity that the given Projectile is let go from P, towards $b$, let another proceed from the fame Point, in a Right-Line paffing directly thro' the Center of Force ; and let the Celerity at U, or the Space that would be uniformly defcribed therewith in 1, the abovefaid Particle of Time, be denoted by $v$: Then, as $a^n$, the Centripetal Force at P, is to $x^n$, that at U, fo is $2r$, the Velocity that might be generated by the former in the given Particle of Time, to $\frac{2rx^n}{a^n}$, that which would be generated by the latter in the fame Time: Wherefore, as 1, that Time, to $\frac{2rx^n}{a^n}$, fo is $\frac{\dot{x}}{v}$, the Time of defcribing U $v$, to $\dot{v}$, the Velocity acquired in this Time: Whence, by multiplying Means and Extremes, &c. we get $v\dot{v} = \frac{2rx^n\dot{x}}{a^n}$, and therefore $vv = \frac{-rx^{n+1}}{\overline{n+1}\times a^n}$ + fome conftant Quantity $d$; which to determine, let U coincide with P, $x$ be $= a$, and $v = m$, and the Equation becomes $\frac{m^2}{2} = \frac{-2ra^{n+1}}{\overline{n+1}\times a^n} + d$; hence $d = \frac{m^2}{2} + \frac{2ra}{n+1}$; which Value being fubftituted above, we fhall have $\frac{U^2}{2} = \frac{m^2}{2} + \frac{2ra}{n+1} - \frac{2rx^{n+1}}{\overline{n+1}\times a^n}$; and therefore U $= \overline{m^2 + \frac{4ra}{n+1} - \frac{4rx^{n+1}}{\overline{n+1}\times a^n}}\Big|^{\frac{1}{2}}$: But this is likewife the Celerity of the firft Projectile at R: For fince both Bodies have the fame Velocity at P, their Velocities, at all equal Diftances from the Centre muft be equal; and therefore $\sqrt{m^2 + \frac{4ra}{n+1} - \frac{4rx^{n+1}}{\overline{n+1}\times a^n}}$, or its

its Equal $\frac{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{v}$ will consequently be the Time of the said given Projectiles moving thro' R $r$, or of describing the the Area $\frac{x\dot{y}}{2}=$ R C$r$, by Radii drawn to the Centre of Force: Wherefore, since $\frac{sma}{2}$ is the Area of the little Triangle PC$b$ that might be uniformly described in, 1, the given Particle of Time, with the Velocity at P; and, because the Areas are as the Times, it will be, as $\frac{sma}{2}$ to 1 (the said Time), so is $\frac{x\dot{y}}{2}$, to $\frac{\sqrt{\dot{x}^2+\dot{y}^2}}{v}$: Hence we get $\dot{y}=\frac{sma\dot{x}}{\sqrt{a^2v^2-m^2s^2a^2}}$, or,

$$\frac{sma\dot{x}}{\sqrt{m^2x^2+\frac{4rax^2}{n+1}-m^2s^2a^2-\frac{4rx^{n+3}}{\overline{n+1}\times a^n}}},$$

by substituting instead of $v^2$, its known Value, as above found. But as C$r$, is to $rn$,

$$:: Cf\ (=a):\ \frac{smaa\dot{x}}{x\sqrt{m^2x^2+\frac{4rax^2}{n+1}-m^2s^2a^2-\frac{4rx^{n+3}}{\overline{n+1}\times a^n}}}=\dot{A}\ (=ef);$$

whose Fluent P$e$ is the Measure of the angular Motion; from which, when found, the Orbit may readily be constructed; because, when P$e$, or the Angle PCR, is given, as well as CR, the Position of the Point R is also given: But this Value of $\dot{A}$ is indeed too much compounded to admit of a Fluent in general Terms, or even by the Quadrature of the Conic Sections, except in certain particular Cases, as where $n$ is equal to 1, —2, —3, or —5, or the Law of centripetal Force, as the first Power of the Distance directly, or the 2[d], 3[d], or 5[th] Powers thereof inversely; therefore, in other Cases, can only be had by infinite Series, &c. or Curves of a superior Order. *Q. E. I.*

 COROL.

## COROL. I.

IF, instead of the absolute Celerity of the Projectile at P, the Ratio thereof to that which it should have to describe the Circle $Pe$, be given, as $p$ to 1, and not only the same Thing, but the Ratio between the Celerity at any other Distance CR, and that which a Body must have to describe a circular Orbit at that Distance, be required: It will be, as $a^n$, the centripetal Force at P, to $x^n$, that at R (or U), so is $r$, the Distance a Body would freely descend by the former of these Forces in 1, the given Particle of Time, to $\frac{rx^n}{a^n}$, that which it would descend by the latter in the same Time: Therefore, if U$s$ be taken equal to $\frac{rx^n}{a^n}$, and $st$ be made perpendicular to AC, it is manifest, that U$t$, being indefinitely small, will be the Distance which a Body must move over in the aforesaid Particle of Time, to describe the Circle UR: But U$t$, by the Property of the Circle, is in that Circumstance $= \sqrt{\frac{2rx^{n+1}}{a^n}}$; wherefore we have, as $\sqrt{\frac{2rx^{n+1}}{a^n}}$, to $v$, or its Equal, $\sqrt{m^2+\frac{4ra}{n+1}-\frac{4rx^{n+1}}{n+1}}$ (above found), so is the Velocity a Body must have to describe that Circle, to that with which the given Projectile arrives at R: Therefore, when $x$ is $=a$, and R coincides with P, the Proportion of $\sqrt{\frac{2rx^{n+1}}{a^n}}$, to $\sqrt{m^2+\frac{4ra}{n+1}-\frac{4rx^{n+1}}{\overline{n+1}\times a^n}}$, which there is, as $\sqrt{2rn}$, to $m$, is given as 1, to $p$, by Supposition; whence, multiplying Ex-

tremes

tremes and Means, we get $m = p\sqrt{2ra}$; which being substituted instead of $m$, in the Value of $v$, it will become $\sqrt{2p^2ra + \frac{4ra}{n+1} - \frac{4rx^{n+1}}{\overline{n+1} \times a^n}}$ or $\overline{p^2 + \frac{2}{n+1} - \frac{2}{n+1} \times \frac{x^{n+1}}{a^{n+1}}}\Big|^{\frac{1}{2}} \times \overline{2ra}\Big|^{\frac{1}{2}}$;

therefore this divided by $\sqrt{\frac{2rx^{n+1}}{a^n}}$, is $\sqrt{\overline{p^2 + \frac{2}{n+1}} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$, for the Ratio that was to be found. And, in like manner, by substituting for $m$ in the Value of $\dot{A}$, we get

$$\frac{spa\,q\dot{x}}{x\sqrt{\overline{p^2 + \frac{2}{n+1}} \times x^2 - p^2s^2a^2 - \frac{2x^{n+3}}{\overline{n+1} \times a^{n+1}}}}$$

for the other Quantity required.

## C O R O L. II.

HENCE, if the Angle $CPb$ be supposed to be diminished *in infinitum*, and $\overline{p^2 + \frac{2}{n+1} - \frac{2}{n+1} \times \frac{x^{n+1}}{a^{n+1}}}\Big|^{\frac{1}{2}} \times \overline{2ra}\Big|^{\frac{1}{2}}$, the said Value of $v$, be taken $= 0$, we shall have

$x = \overline{\tfrac{1}{2}pp \times \overline{n+1} + 1}\Big|^{\frac{1}{n+1}} \times a$, $(= CA)$ the Height to which the Body would ascend, if projected directly upwards; therefore, $\overline{\tfrac{1}{2}pp \times \overline{n+1} + 1}\Big|^{\frac{1}{n+1}} \times a - a = AP$, is the Distance it must freely descend to acquire the given Velocity; which Distance, therefore, with an uniform Centripetal Force, where $n = 0$, will be $= \frac{ppa}{2}$; and with a Force inversely as the Square of the Distance, $= \frac{ppa}{2 - pp}$. But when $p$ is $= 1$, or the Velocity of the Projectile at P is just sufficient

to

to retain a Body in the circular Orbit P $e$, A P then becomes $\overline{\frac{3+n}{2}}\Big|^{\frac{1}{n+1}} \times a - a$; which in the said two Cases, will be $\frac{1}{2}a$, and $a$ respectively; but infinite when $n$ is $=-3$.

## COROL. III.

WHEN $n+1$ is a positive Number, the Velocity $\sqrt{2ra} \times \overline{p^2 + \frac{2}{n+1} - \frac{2}{n+1} \times \frac{x^{n+1}}{a^{n+1}}}\Big|^{\frac{1}{2}}$, at the Centre C, where $x$ becomes $=o$, will, it appears, be barely equal to $\sqrt{2ra \times \overline{p^2 + \frac{2}{n+1}}}$; but, when $n+1$ is negative, or the Law of Centripetal Force more than the first Power of the Distance inversely, it will be infinite; because then, the Index being negative, $x^{n+1}$ (or its Equal $o^{n+1}$) will come into the Denominator.

## COROL. IV.

MOREOVER, when $n+1$ is negative, and $x$ infinite, the said Velocity will also become $\sqrt{2ra \times \overline{p^2 + \frac{2}{n+1}}}$; because then, for the Reason above specified, $x^{n+1}$ will be $=o$: And therefore, when the Centripetal Force is more than the first Power of the Distance inversely, a Projectile moving from P with the given Velocity $p\sqrt{2ar}$ ($=m$) along the Right-Line PA, will ascend even to an infinite Height, and have a Velocity there signified by $\sqrt{ra \times \overline{p^2 + \frac{2}{n+1}}}$, or in Proportion to the given Velocity, as $\sqrt{p^2 + \frac{2}{n+1}}$, to $p$, provided $p^2 + \frac{2}{n+1}$ be

be positive; for otherwise the Thing is impossible, the Square Root of that Quantity being manifestly so.

## COROL. V.

HENCE, if the least Velocity that can carry the Body to an infinite Height, or that which it would acquire by freely descending from the same Height, be required: By making $p^2 + \frac{-2}{n+1} = o$, we shall have $p = \sqrt{\frac{-2}{n+1}}$; which, substituted in $p\sqrt{2ar}$ gives $\sqrt{\frac{-2}{n+1}} \times \sqrt{2ar} = 2\sqrt{\frac{-ra}{n+1}}$, for the Value sought; and this, it is manifest, is to $\sqrt{2ar}$, the Velocity a Body must have to describe the Circle P$e$, as $\sqrt{\frac{-2}{n+1}}$, to Unity: Therefore, when $n$ is less than $-3$, or the Law of Centripetal Force more than the Cube of the Distance inversely, a less Velocity will carry a Projectile to an infinite Height in a Right-Line, than can retain it in a circular Orbit, was it turned into a proper Direction.

## COROL. VI.

WHEREFORE, if it were required, how far a Body must descend by an uniform Force equal to that affecting the Projectile at the Point P, to acquire the same Celerity that another Body, by freely falling from an infinite Height (as above) has at its Arrival to that Point; then, by substituting the Value $\sqrt{\frac{-2}{n+1}}$, as found in the last Article, instead of its Equal, in $\frac{ppa}{2}$ (see *Cor.* II.) there comes out $\frac{-a}{n+1}$ for the Value sought: And hence it appears, that the Velocity with which a Body, falling freely from an in-

finite Height, would impinge on the Earth, is no greater than that which another Body may acquire by an uniform Gravity, equal to that at its Surface, in falling freely thro' a Space equal to its Semi-diameter.

## SCHOLIUM.

FROM the Ratio found in *Corollary* I. between the Velocity with which the Body arrives at any Diſtance ($x$) from the Centre of Force, and that which it ought to have to deſcribe a Circle at the ſame Diſtance, it will not be difficult to determine in what Caſes the Body will be compelled to fall to the Centre, and in what other Caſes it will fly *ad infinitum* therefrom. For, firſt, if the Body in moving from P, begins to deſcend, or the Angle C P $b$ be acute, I ſay, it will continue to do ſo 'till it actually falls into the Centre of Force, if the Quantity $\left( \sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}} \right)$ in its Acceſs thereto, be not ſomewhere greater than Unity; or, which is the ſame in effect, unleſs the Body has ſomewhere a Velocity more than ſufficient to retain it in a circular Orbit at its own Diſtance from the Center of Force: For, if it ever begins to aſcend, it muſt be at a Point, as D, where a Right-Line, drawn from the Centre, cuts the Orbit perpendicularly, and there, it is manifeſt, the Celerity muſt be as above ſpecified, otherwiſe the Body will ſtill continue to deſcend, or elſe move in the Circle D L about the Center C, which is equally abſurd. On the contrary, if the ſaid Quantity, in approaching the Centre, increaſes ſo as to become greater than Unity, or be every where ſo; then, the Velocity at all inferior Diſtances, being greater than the Velocity that is ſufficient to retain a Body in a circular Orbit at any ſuch Diſtance, the Projectile cannot, it is evident, be forced to the Centre.

But

But, on the other hand, the Angle C P $b$, being fuppofed obtufe, it will evidently appear from a like Reafoning, that, if the faid Quantity be always greater than Unity, or the Body in its Recefs from the Center, has, in every Place thro' which it paffeth, a Velocity greater than is fufficient to retain it in a circular Orbit at the Diftance of that Place from the Center of Force, it muft, of confequence, continue to afcend *ad infinitum*.

Now, therefore, to find in what Laws of Centripetal Force thefe different Cafes obtain, let the Angle C P $b$ be firft fuppofed acute, or the Body moving towards the Centre, and $x$ in the abovefaid Quantity $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$, to be infinitely fmall; then it is evident, that that Quantity will become either $\sqrt{\frac{-2}{n+1}}$, or *infinite*, according as $n+1$ is a negative Number, or otherwife; wherefore, in the latter of thefe two Cafes, the Body can never be forced into the Centre; neither can it in the former, when $n$ has any Value betwixt $-1$ and $-3$, as is manifeft from above, becaufe $\sqrt{\frac{-2}{n+1}}$ is greater than Unity (Rectilinear Motion being here excepted:) Nor will either of thefe Conclufions hold lefs true, when the Angle C P $b$ is obtufe; for it is obvious, that if the Projectile cannot be forced to the Centre, when directed towards it with the leaft Obliquity, it never can, when the Obliquity is increafed: But on the contrary, if $n+1$ be either equal to or lefs than $-2$, and $p$ be lefs than $1$; then the faid Value $\sqrt{\frac{-2}{n+1}}$ not being greater than Unity, the Projectile muft inevitably be drawn into the Centre; for, the afore-mentioned general Expreffion not exceeding

exceeding Unity, neither at the given Distance $a$, nor at the least assignable Distance, cannot at an intermediate Distance; because, in the Descent of the Body, the Expression must either increase or decrease continually, there being only one Dimension of the variable Quantity ($x$) concerned. But, when $p$ is greater than Unity, other Things continuing the same, I say, the Body, if it escapes the Centre, and once begins to ascend, it will continue to fly from the same *ad infinitum*, For, since the Part D L, &c. of the Trajectory, which it will begin to describe on its leaving the lowest Point D, is in every respect equal and similar to D R, &c. if another Body projected upwards from P, in the opposite Direction, with the same Velocity, continues to ascend *ad infinitum*, our first Projectile, after it has passed the lowest Point, must do so too, and *vice versa*; therefore $p^2 + \frac{2}{n+1}$ being there affirmative, and the Angle CP$b$ obtuse, the Quantity $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$, when $x$ is infinite, will also be infinite; whence from the above Reasoning, the Position is manifest. Hence we conclude, *first*, that when $n$ is greater than $-3$, or the Law of Centripetal Force, as any Power of the Distance directly, or less than the Cube thereof inversely, the Body cannot possibly fall into the Centre, except in a Right-Line. And, *secondly*, that, when the Force is, as the Cube, or more than the Cube of the Distance inversely, it must either be forced to the Centre, or fly an infinite Distance therefrom, unless it moves in a Circle.

Furthermore, becauset he abovesaid Quantity, when $x$ is infinite, in all Cases where $n+1$ is negative, and $pp$ greater than $\frac{-2}{n+1}$, appears to be greater than Unity, it follows, that in all

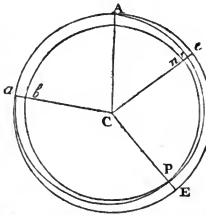
thofe

thofe Cafes, the Body may afcend, even to an infinite Height, and actually will do fo, when $n$ has any Value betwixt $-1$ and $-3$; becaufe then, tho' the Body fhould at firft approach towards the Centre, its Afcent cannot be anticipated by being drawn into it, as it may, when the Value of $n$ is fmaller, as has been above fhewn.

*Note*, The fame Things may be otherwife determined by Help of the laft general Value of $\dot{A}$; for if $\overline{pp + \frac{2}{\overline{n+1}} \times xx - p^2 s^2 a^2 - \frac{2x^{n+3}}{\overline{n+1} \times a^{\overline{n+1}}} \times xx}$, the Square of its Divifor be made equal to nothing, the affirmative Roots of that Equation, or Values of $x$, will give the greateft and leaft Diftances of the Projectile from the Centre of Force, and therefore in thofe Cafes, where it is found not to admit of two fuch Roots, the Body muft either fall into the Centre, or fly it *ad infinitum*.

## PROPOSITION III.

*To find the Motion, or Angular Distance of the Apsides, in Orbits nearly circular; the centripetal Force being as any Power of the Distance.*




LET A $r$ P $a$ be the proposed Orbit, A and P two Places of the higher and lower *Apsides*, A $e$ E $a$ A, and $n$ P $b$ $n$, Circles described with the Radii A C, C P about C, the Centre of Force; let $r$ be a Point in the Trajectory taken at Pleasure; and let the Velocity of the Body at the higher *Apse* be to that which it ought to have to retain itself in the Circle A $e$ E, as $\sqrt{1-e}$, to 1; calling AC, 1; $re$, $y$; C$r$, $1-y$; and A$e$, A: Then, by substituting, 1 for $a$, 1 for $s$, $\sqrt{1-e}$ for $p$, $1-y$ for $x$, and $\dot{y}$ for $\dot{x}$, in $\dfrac{spaa\dot{x}}{x\sqrt{p^2+\frac{2}{n+1}\times xx-p^2s^2a^2-\frac{2x^{n+3}}{n+1\times a^{n+1}}}}$ the general Value of $\dot{A}$, as found by the last *Prob.* the same it is manifest, will become $\dfrac{\sqrt{1-e}\times\dot{y}}{\overline{1-y}\sqrt{1-e+\frac{2}{n+1}\times\overline{1-y}|^2-1+e-\frac{2\times\overline{1-y}|^{n+3}}{n+1}}}$ for the Value of $\dot{A}$ in this particular Case; which by reducing

cing $\overline{1-y}|^2$ and $\overline{1-y}|^{n+3}$ into ſimple Terms, is

$$\frac{\dot{y}\sqrt{1-e}}{1-y\sqrt{1-e+\frac{2}{n+1}\times\overline{1-2y+yy}-1+e-\frac{2}{n+1}\times 1-n\overline{+3}\times y+n+3\times\frac{n+2}{2}y^2}}\text{, \&c.}$$

$$\text{\&c.} = \frac{\dot{y}\sqrt{1-e}}{\overline{1-y}\sqrt{2ey-ey^2-\overline{n+3}\times yy-\frac{2}{n+1}\times\frac{n+3}{1}\times\frac{n+2}{2}\times\frac{n+1}{3}y^3\text{, \&c.}}}$$

but, becauſe $e$ and $y$, by the Nature of the Queſtion, are very ſmall, all the Terms wherein more than two Dimenſions of theſe Quantities are concerned, may be rejected as inconſiderable in reſpect of the reſt; by doing which, our Equation becomes $\dot{A} = \frac{\dot{y}\sqrt{1-e}}{1-y\sqrt{2ey-\overline{n+3}\times yy}}$; which (for the above Reaſon) is $= \frac{\dot{y}}{\sqrt{2ey-\overline{n+3}\times yy}}$, or $\frac{1}{\sqrt{n+3}}\times\frac{\dot{y}}{\sqrt{\frac{2ey}{n+3}-yy}}$ very nearly: But the Fluent of $\frac{\dot{y}}{\sqrt{\frac{2ey}{n+3}-yy}}$ when $\sqrt{\frac{2ey}{n+3}-yy}$ becomes $=0$, or $A=AeE$, is equal to a Semi-circle whoſe Radius is Unity, or to 180 Degrees; therefore $\frac{1}{\sqrt{n+3}}\times 180^\circ = \frac{180}{\sqrt{n+3}}$ Degrees, is the Meaſure of the Angle ACP. *Q. E. I.*

## COROL. I.

WHEN $n$ is equal to, or leſs than $-3$, then the Value $\left(\frac{180}{\sqrt{n+3}}\right)$ of the Angle ACP, becoming either infinite or impoſſible, it follows, that if the Law of Centripetal Force be, as the Cube, or more than the Cube of the Diſtance inverſely,

inverſely, the Trajectory cannot have more than one *Apſide:* And, therefore, the Projectile in all ſuch Caſes muſt inevitably either fall into the Centre of Force, or fly from it *ad infinitum*, unleſs it moves in a Circle; which is agreeable to the *Scholium* aforegoing. But, if $n$ be equal to 1, 0, $-1$, or $-2$; then will the Angular Diſtance of the two *Apſides* be, 90°: 00′, 103°: 55′, 127°: 17′, or 180°: 00′, reſpectively; the firſt and laſt of which we are aſſur'd of from other Principles.

## COROL. II.

IF the Diſtance (D) of the *Apſides* be given, and the Law of Centripetal Force from thence be required: Then, by making $\frac{180}{\sqrt{n+3}}$ equal to D, we ſhall have $\overline{\frac{180}{D}}\Big|^2 - 3, = n$, for the Value ſought: Hence, if D be 360°, or the Body takes up one intire Revolution in going from one *Apſe* to the other; then, muſt the Law of Centripetal Force be reciprocally as that Power of the Diſtance, whoſe Exponent is $2\frac{3}{4}$; but, if either *Apſe*, from the Time of the Body leaving it, to its Return again, has mov'd forward only a very ſmall Diſtance, E, or D be $= 180° + \frac{E}{2}$, the Force will then be inverſely as the $2 + \frac{E}{180}$ Power of the Diſtance, very nearly.

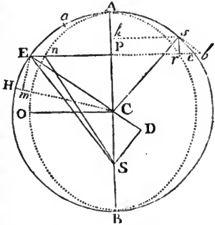
## SCHOLIUM.

IF $x$ be any Diſtance of the Projectile from the Centre of Force, and the Law, by which it tends towards the ſame Centre, be every where, as $cx^n + dx^m + ex^p + fx^q$, &c. $c$, $d$, &c. $n$, $m$, &c. being determinate Quantities; and if

if $a$ be the Diſtance of one of the *Apſides* from that Centre, the angular Diſtance of thoſe *Apſides* will be

$$\overline{\frac{ca^n + da^m + ea^p + fa^q, \&c.}{\overline{3+n}\times ca^n + \overline{3+m}\times da^m + \overline{3+p}\times ea^p}}\Bigg|^{\frac{1}{2}} \times 180°.$$

## *From the* MEAN ANOMALY *of a Planet given; to find its* PLACE *in its* ORBIT.




LET AOB be the given Orbit, S the Sun in one of the Foci, AC the Semi-Tranſverſe Axis, CO the Semi-Conjugate, AEHBA a Circle circumſcribing the Ellipſis, and let $n$ be the Place of the Planet at any given Time after or before its paſſing, A, the Aphelion; thro' which draw E$n$P perpendicular to AB, and having joined the Points E S, E C, S$n$, and made SD perpendicular to ECD, take the Arch EH equal to SD, and the Arch A$a$ equal to SC. Then, the Sector ECH being equal to the Triangle ECS, ACHA will be equal to ASEA; inaſmuch as the former of thoſe Areas is compounded of the Sector ACE and ECH, and the latter of the ſame Sector and the Triangle ECS: Wherefore, ſince the Area ASEA, is to AEBCA, half the Circle, as the Elliptical Area A$n$SA, to the Semi-Ellipſis A$n$BCA, by a known Relation of the two Curves; if,

M instead

inftead of A S E A, its Equal be fubftituted, we fhall have, as A C H A : A E B A :: A $n$ S A : A $n$ B A; but ACHA is to A E B A, as the Arch A H to A E B, the Semi-Circumference; and therefore it will be, as A $n$ B A : A $n$ S A :: A E B : A H: Wherefore fince the Areas A$n$BA, A$n$SA, defcribed by Radii drawn to S, the Center of Force, are as the Times of their Defcription, it will be, as the Time of defcribing A $n$ B A, or that of Half one Revolution, is to the given Time of defcribing A $n$ S A, fo is A E B to A H; which, therefore, is the given Mean Anomaly in this Pofition, or the Arch proportional to the Time of the Planet's moving thro' A $n$.

Let now $AC = 1$, $CS = e$, $AH = D$, A E equal E, its Sine E P equal $x$, and its Co-fine $CP = y$. Then, from the Similarity of the Triangles C E P, C S D, we fhall have, EC : EP :: A$a$ (SC) : EH (SD) and, confequently, $AE + \frac{EP}{EC} \times Aa = AH$, or $E + x \times Aa$, $= D$; which Equation, it is manifeft, will hold equally, whether the Arches A E, A$a$, and A H, be taken in Degrees or in Parts of the Radius: But now, in order to folve the fame, let the required Arch, or Value of E, be eftimated pretty near the Truth, and let this affumed Value be denoted by $As = \acute{E}$, its Difference ($se$) from the Truth, by $\dot{E}$; and $D - As - \frac{Rs}{AC} \times Aa = D - \acute{E} - \acute{x} \times Aa$, the Error of the the Equation, by $\acute{R}$; make $vr$ parallel to A B, and let $sb$ be a Tangent to the Circle at the Point $s$: Then, as $sb$, by reafon of its Smallnefs, may, in this Cafe, be confidered as equal to $es$, and becaufe of the Similarity of the Triangles C$ks$, $srb$,

we ſhall have as $1\ (Cs) : \acute{y}\ (Ck) :: \dot{E} : \acute{y} \times \dot{E} = rb$, or $er$, very nearly; whence $E = \acute{E} + \dot{E}$, and $PE = x = \acute{x} + \acute{y} \times \dot{E}$, which Values therefore being ſubſtituted in the general Equation $E + x \times Aa = D$, there comes out $\acute{E} + \dot{E} + \acute{x} \times Aa + \acute{y} \times \dot{E} \times Aa = D$ very nearly; wherefore $\dot{E} = \left(\frac{D - \acute{E} - \acute{x} \times Aa}{1 + \acute{y} \times Aa} = \right) \frac{D - \acute{E} - \acute{x} \times Aa}{1 + e\acute{y}}$ or, $= \frac{\acute{R}}{1 + e\acute{y}}$ nearly: Hence it appears, that, if the Error of the Equation be divided by $1 + e\acute{y}$, and the Quotient added to, or ſubtracted from the firſt or aſſumed Value of E, there will ariſe a new Value of that Quantity much nearer the Truth than the former: And if with this new Value, and thoſe of $x$ and $y$ correſponding thereto, we proceed to a new Error, or compute the Value of R, and that of the Diviſor $1 + ey$, *&c.* it is likewiſe evident, for the very ſame Reaſons, that a third Value of E may be found, by the ſame Theorem, ſtill nearer the Truth than the preceding, and from thence another, and ſo another, *&c.* 'till we arrive to any Accuracy deſired, each Operation, at leaſt, doubling the Number of Places; ſo that in the moſt excentric of the planetary Orbits two Operations will be found ſufficient to bring out the Angle ACE to leſs than a Second: And when that is known, as EP and SP are then given, the Angle $nSP$ may be eaſily had; for, by the Property of Curve, it is $AC : CO :: EP : Pn = \frac{CO \times EP}{AC}$, and $SP : \frac{EP \times CO}{AC}\ (Pn) :: AC$ (Radius) $: \frac{EP \times CO}{SP} =$ the Tangent of $ASn$. *Q. E. I.*

Otherwiſe,

Otherwise,

Let Radius $EC = r$ and the general Equation $AE + \frac{EP}{EC} \times Aa = AH$, or $E = D - \frac{x}{r} \times Aa$ be again resumed; then, the Orbit not being very Excentric, E will, it is evident, be nearly equal to D; and, consequently, ($x$) the Sine of E, nearly equal to the Sine of D: Therefore, if the Sine of D be substituted for $x$, and the said Sine be denoted by $\acute{x}$ (signifying the first Value of $x$) it is obvious, that $D - \frac{\acute{x}}{r} \times Aa$ will be nearer to the true Value of E, than D, and, consequently, that the Sine of $D - \frac{\acute{x}}{r} \times Aa$ (which I call $x''$) nearer to $x$ than ($\acute{x}$) the Sine of D; wherefore $D - \frac{x''}{r} \times Aa$, must be, still, nearer the Truth, or the required Value of E, than $D - \frac{\acute{x}}{r} \times Aa$, and, consequently, its Sine (which I call $x'''$) still, nearer $x$, than ($x''$) the Sine of $D - \frac{\acute{x}}{r} \times Aa$: In like manner, the Sine of $D - \frac{x'''}{r} \times Aa$ (or $x''''$) will appear to be nearer $x$ than $x'''$, and $D - \frac{x''''}{r} \times Aa$, nearer to the required Value than $D - \frac{x'''}{r} \times Aa$, &c. &c. Whence the following Method of Solution is manifest.

Let 1.758123, the Log. of (57.2958) the Number of Degrees in an Arch equal in Length to Radius, be added to the Logarithm

Logarithm of the Excentricity, and from the Sum deduct the Logarithm of Half the greater Axis; the Remainder will be a 4th Logarithm (L); which, being once computed, will ſerve in all Caſes of that Orbit: To this Logarithm add the Logarithmical Sine of the given Mean Anomaly reckoned to or from the Aphelion; the Sum, rejecting Radius, will be the Logarithm of an Arch in Degrees; which, being taken from the Mean Anomaly, and the Sine of the Remainder added to the ſaid Logarithm, the Sum, rejecting Radius, will be the Logarithm of a 2d Arch; which, in like manner, being taken from the Mean Anomaly, and the Sine of the Remainder added to the ſame Logarithm, the Sum, rejecting Radius, will be the Logarithm of a 3d Arch; from whence, by repeating the Operation in the very ſame manner, a 4th Arch will be found, and ſo a 5th, *&c.* 'till we arrive to any aſſigned Exactneſs; the Error in the *Anomaly Excentri*, or Angle ACE, which Angle is to be expreſſed by the Difference of the Mean Anomaly and the laſt of the ſaid Arches, being always much leſs than the Difference of the ſaid Arch and that which immediately precedes it, from which Angle the true Anomaly is had as in the above Caſe. *Q. E. I.*

Otherwiſe,

The foregoing Conſtruction being retained, let Radius (AC) $=1$, the Sine of the given *Anomaly* ACH $=a$, its Co-ſine $=b$, and let E$m$ be the Sine of E H: Then will $a$ C$m - b \times$ E$m$ $= x$, the Sine of the Difference of thoſe Angles, by the Elements of *Trigonometry*; but EH being $=ex$, E$m$ (by the ſame) will be $ex - \frac{e^3x^3}{2.3} + \frac{e^5x^5}{2.3.4.5}$, *&c.* and C$m = 1 - \frac{e^2x^2}{2} + \frac{e^4x^4}{2\,3\,4}$, *&c.* whence, by Subſtitution, *&c.* we get $\overline{1+be} \times x$

N +

$+\frac{ae^2x^2}{2}-\frac{be^3x^3}{6}-\frac{ae^4x^4}{24}$, &c. $=a$, where, by inverting the Series, $x$ comes out $=\frac{a}{1+be}-\frac{\frac{1}{2}a^2e^2}{\overline{1+be}|^3}+\frac{a^3}{\overline{1+be}|^3}\times\overline{\frac{\frac{1}{6}be^3}{1+be}}+\overline{\frac{\frac{1}{2}a^2e^2}{\overline{1+be}|^2}}$, &c. $=a\times\overline{1-be+\frac{2bb-aa}{2}\times ee+\frac{5aa-3bb}{3}\times be^3}$ &c. from whence, if the Excentricity be not very large, the Angle AS$n$ may be had as above to any Degree of Exactneſs, *Q. E. I.*

*Note*, That, in theſe Solutions, when E is greater than a Right Angle, its Co-ſine $y$ is to be conſidered as a negative Quantity.

## SCHOLIUM.

As the foregoing Methods of Solution may ſeem tedious or perplexed for common Practice, a ſhort Approximation, tho' limited in Point of Exactneſs, may be of Service. In order to this, we have have given EP $(=x)=a\times\overline{1-be+\frac{2bb-aa}{2}\times ee+\frac{5aa-3bb}{3}\times be^3}$, &c. as above, from whence PC $(=y=\sqrt{\overline{1-xx}})$ is $=b+a^2e-\frac{3ba^2e^2}{2}+\frac{6bb-2aa}{3}\times a^2e^3+\frac{65a^2-6bb}{24}\times ba^2e^4$, &c. But, by the Nature of the Curve $1+ey$ is $=$S$n$, and P$n=\sqrt{\overline{1-ee}}\times x$ $\left(=\frac{\text{OC}\times\text{EP}}{\text{AC}}\right)$ and therefore $\frac{x\sqrt{\overline{1-ee}}}{1+ey}$ $\left(=\frac{\text{AC}\times\text{P}n}{\text{S}n}\right)$ will be the Sine of the Angle AS$n$, or true *Anomaly*, and $\frac{e+y}{1+ey}$ $\left(=\frac{\text{AC}\times\text{SP}}{\text{S}n}\right)$ its Co-ſine: Hence, by the Elements of Trigonometry $a\times\frac{e+y}{1+ey}-b\times\frac{x\sqrt{\overline{1-ee}}}{1+ey}=\frac{ae+ay-bx\sqrt{\overline{1-ee}}}{1+ey}$ will be the Sine of the Difference of the Angles HCA, AS$n$, or of the Equation of the Orbit; wherein, by ſubſtituting, inſtead

instead of $x$ and $y$, their respective Values, and contracting the whole by Division, &c. there will come out $2ae$ into $1 - \frac{5be}{4} + \overline{\frac{3bb}{2} - \frac{4aa}{3}} \times ee + \overline{\frac{245aa}{10} - \frac{27bb}{16}} \times be^3$, &c. which, when $e$ is not very large, will appear to be equal to $\frac{2ae}{1+\frac{5be}{4}} - \frac{1}{3} \times \overline{\frac{2ae}{1+\frac{5be}{4}}}\Big|^3$ very nearly; for this, converted to a Series, is $2ae - \frac{5abe^2}{2} + \frac{25ab^2e^3}{8} - \frac{8a^3e^3}{3} + 10a^3be^4 - \frac{125ab^3e^4}{32}$, &c. from which, if the former Series be taken, there will remain only $\frac{abbe^3}{8} + \overline{\frac{17bb}{32} - \frac{5aa}{24}} \times be^4$, &c. Hence is deduced the following

## PRACTICAL RULE

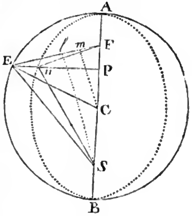
### For finding the Equation of the Centre from the Mean Anomaly given.

*As Radius, to the Co-sine of the given Anomaly, so is $\frac{5}{4}$ Parts of the Excentricity of the Orbit, to a fourth Number; which Number add to half the greater Axis, if the Anomaly be less than* 90, *or more than* 270 *Degrees, otherwise subtract from the same: Say, as the Sum or Remainder, is to Double the Excentricity, so is the (Logarithmic) Sine of the given Anomaly, to the Sine of a first Arch; from three Times which Sine deduct the double Radius, the Remainder will be the Sine of a second Arch, whose $\frac{1}{3}$ Part, taken from the former, leaves the Equation sought.*

And

And it muſt be noted, that this *Rule*, in the Orbits of *Saturn*, *Jupiter*, and the *Moon*, anſwers to a Second, and in thoſe of the *Earth* and *Venus* to leſs than $\frac{1}{10}$ of a Second. And, in theſe two laſt, the Arch firſt found will, without farther Correction, be ſufficiently exact to anſwer to the niceſt Obſervations, the Error never amounting to above 2 or 3 Seconds; which is more correct than either the noted Hypotheſis of *Ward* or *Bullialdus*, as will appear from the following Examination of thoſe Hypotheſes, which, as they have been much celebrated, and come near the Truth in many Caſes, may here alſo deſerve a particular Conſideration. And, to begin with the latter, which ſuppoſes the Angle $AFn$ made at F the upper Focus by the Aphelion and ($n$) the Planet to be the Mean Anomaly, and therefore $SnF$ the Equation. Becauſe $a$ the Sine and $b$ the Co-ſine of the ſaid Angle are given, by the Nature of the Ellipſis, $Sn = \frac{1+2eb+ee}{1+be}$ is alſo given; whence, by *Plain Trigonometry*, it will be, as $\frac{1+2eb+ee}{1+be} : a :: 2e$ (SF) $: 2ae \times \frac{1+be}{1+2be+ee}$ equal to the Sine of $SnF$, which, put in a Series, is $2ae \times \overline{1 - eb + \overline{2bb - 1} \times ee + \overline{3 - 4bb} \times be^3}$, &c. and this taken from $2ae \times \overline{1 - \frac{5eb}{4} + \overline{\frac{3bb}{2} - \frac{4aa}{3}} \times ee}$, &c. leaves $2ae \times -\frac{eb}{4} + \overline{\frac{bb}{2} - \frac{aa}{3}} \times ee$, &c. for the Error of this Hypotheſis. But now for the other, where, $EnP$ being perpendicular to AB, AFE is ſuppoſed the Mean Anomaly. Let SC and $Cm$ be perpendicular to FE; then it will be as 1 (EC) : $a$ (the Sine of AFE, or CFE) :: $e$ (CF) : $ae = Cm$, the Sine of CEF; whence $Em$, its

its Co-sine, $=\sqrt{1-a^2e^2}$: Again, as 1 (the Sine of C$m$F) : $e$ (CF) :: $-b$ (the Sine of FC$m$) : $-eb=$ F$m$, equal also to C$m$, because CS is $=$ FC; for which Reason S$l$ is double to $m$C; wherefore it will be, as $\sqrt{1-a^2e^2}+eb$ (E$l$) : $2ae$ (S$l$) :: 1 (Radius) to $\frac{2ae}{\sqrt{1-a^2e^2}+eb}$ = the Tangent of SEF; which, in a Series, will be $2ae\times\overline{1-eb+\frac{a^2e^2}{2}+e^2b^2}$, &c. whence the corresponding Sine is easily found $=2ae\times\overline{1-eb+b^2e^2-\frac{3aaee}{2}}$, &c. and this taken from $2ae\times\overline{1-\frac{5eb}{4}+\overline{\frac{3bb}{2}-\frac{4aa}{3}}}\times e^2$, gives $2ae$ into $-\frac{eb}{4}+\overline{\frac{bb}{2}+\frac{aa}{0}}\times ee$, &c. for the Error in this Case




Hence it will appear, that the greatest Error of each of these Hypotheses, in the Orbit of *Mars*, where $e$ is upwards of .09, will be about 5 or 6 Minutes, and in the other planetary Orbits, according to the Squares of their Excentricities (in Parts of their own Semi-Axis) nearly; it also appears, that towards the Aphelion the Circular Hypothesis will be the more correct, and near the Perihelion the other; and, lastly, that both Hypotheses make the Equation too large in the higher, and too small in the lower, Part of the Orbit.

 Having

Having ſhewn how much theſe two noted Hypotheſes, (by many ſo much eſteemed) differ from Truth, it may be proper to proceed now to give ſome *Examples* of the preceding Methods, whereby the *Problem* is more correctly ſolved.

## EXAMPLE I.

LET the Excentricity of the propoſed Orbit be $\frac{5}{100}$ of the Mean Diſtance or Semi-Tranſverſe Axis, and the given Mean Anomaly $72^{\circ}\ 12'\ 36'' = 72.21$ Degrees; and let the *Anomaly Excentri*, by the firſt Method, be required.

It will be, as 1, the Semi-Tranſverſe, to .05, the Excentricity, ſo is 57.2958, the Number of Deg. in an Arch, equal in Length to Radius, to $2.86479 =$ the Arch A $a$; wherefore, the general Equation, in reſpect to this Orbit, will be $E + 2.86479 \times x = D$, and by writing therein the given Anomaly, inſtead of D, it will, in this particular Caſe, become $E + 2.86479 \times x = 72.21$. Now, becauſe $2.86479 \times x$ muſt be leſs than 2.86479, E, it is evident, can neither be much leſſer, nor much greater, than 70 Degrees; therefore, I eſtimate the ſame at 70 Degrees, and then ſay, as Radius, to the Sine of that Angle, ſo is 2.86479, to 2.692; whence $E - D + 2.86479 \times x$ equal 0.482, which is the Error, or firſt Value of R: Again, for the Diviſor $1 + ey$, as Radius, to ($y$) the Co-ſine of $70^{\circ}$, ſo is .05, ($e$) $.0171 = ey$; therefore $1 + ey = 1.0171$, and $\frac{R}{1 + ey} = \frac{0.482}{1.0171} = 0.464$; which, being taken from $70^{\circ}$, gives 69.536 for the next Value of E; wherefore, it will be, as Radius, to the Sine of 69.536, or $69^{\circ}\ 32'\frac{16}{100}$, ſo is the ſaid Co-efficient 2.86479, to 2.68401; from whence the

next

next Value of R is found equal 0.01001; and this, divided by the next Value of $1+ey$, or even by 1.0171, the last Value, and the Quotient taken from 69.536, leaves the true Value of E ($= 69^\circ\ 31'\ 34''$) to less than a Second.

## EXAMPLE II.

LET the same Things be proposed, as in the preceding *Example*, and the Answer according to the second Method be required.

The Value of L, or the Log. of the Arc A *a*, as found by the last Example, being 457093, I add thereto the Logarithmic Sine of $72^\circ\ 12'\ 36''$, or $72^\circ .21$, the Sum, rejecting Radius, is the Logarithm of 2.73, the first Arch, which subtracted from 72.21, the Remainder will be 69.48; to whose Sine adding the said Value of L, the Sum, deducting Radius, will be the Logarithm of 2.683, the second Arch; with which, repeating the Operation, the third Arch will come out 2.6838, &c. and this taken from 72.21, leaves 69.5261, &c. or $69^\circ\ 31'\ 34''$ for the *Anomaly Excentri*; from whence the *True Anomaly* will come out $66^\circ\ 52'\ 50''$.

## EXAMPLE III.

THE same Things being given; by the practical Rule, it will be, as Radius to the Co-sine of $72^\circ\ 12'\ 36''$, so is .0625 ($=\frac{5}{4}$ of .05) to .0191; again, as $1+.0191$, to 0.1, the double Excentricity, so is the Sine of the same Angle, to the Sine of $5^\circ\ 21'\ 41''$; three times whose Log. Sine, *minus* double the Radius, is the Sine of $2'\ 48''$; the $\frac{1}{3}$ Part whereof being taken from $5^\circ\ 21'\ 41''$, leaves $5^\circ\ 20'\ 45''$ for the Equation of the Center, and ∴ this taken from $72\ 12'\ 36''$ will give $66^\circ\ 52'\ 51''$, equal to the *True* Anomaly very nearly.

Of

## Of the Motion of Projectiles in resisting Mediums.

### PROPOSITION I.

*Supposing that a Body, let go from a given Point, with a given Velocity, directly to or from a Centre, towards which it uniformly gravitates, is resisted by a similar Medium, in the Ratio of certain Powers of the Velocity, whose Indices are represented by the given Numbers, r, s, t,* &c. *And supposing the Part of the whole Resistance, at the said given Point, corresponding to each of those Powers, as well as the Force of Gravity, to be given; 'tis required to find the Relation of the Times, the Velocities, and the Spaces gone over.*


LET P be the given Point, DPC the Right Line in which the Body moves, and D, *e*, any two Points therein indefinitely near to each other: Suppose the Velocity at P to be sufficient to carry the Body, uniformly, over a given Distance *g*, in a given Time *b*; and let *m* be the Space, which would be described in the same Time with the Velocity, that would be generated in that Time *in vacuo* by a Force equal to the Body's specifick Gravity in the given Medium; let the Part of the Resistance, which is as the *r* Power of the Celerity, at the aforesaid Point, be such, that the Body in moving over a given Distance *b*, with its Velocity uniformly continued, would from that Part alone, meet with a Resistance sufficient to take away its whole Motion; or which is the same, let *b* be

2 the

the Diſtance that might be deſcribed with the Velocity at P in the Time that the Body would, by the ſaid Part alone, have all its Motion deſtroyed, was the Reſiſtance to continue the ſame as at the firſt Inſtant; and let the like Diſtances, with reſpect to the other Parts of the Reſiſtance, that are as the Powers of the Celerity, whoſe Indices are, $s$, $t$, &c. be $c$, $d$, &c. reſpectively; laſtly, let $PD = x$, $De = Pq = \dot{x}$, the Time of deſcribing $PD = T$, and the Space the Body would move over in the given Time $b$, with the Velocity at D, $= v$.

Then it will be, as $g : b :: \dot{x}\ (Pq) : \frac{b\dot{x}}{g}$ the Time of deſcribing $Pq$, and as $b : g :: \dot{x}\ (Pq) : \frac{r\dot{x}}{b}$, the Velocity deſtroyed by that Part of the Reſiſtance, which is as the $r$ Power of the Celerity, in that Time; therefore, the Velocity at D being to the Velocity at P, as $v$ to $g$, that deſtroyed, by the ſame Part, in the ſame Time, from the Body's leaving D, will conſequently be $\frac{g\dot{x}}{b} \times \frac{v^r}{g^r} = \frac{v^r\dot{x}}{bg^{r-1}}$, becauſe this Part of Reſiſtance is as the $r$ Power of the Velocity: But the Time of deſcribing $De$, is to the Time of deſcribing $Pq$, as $g$ to $v$; therefore the Reſiſtance ariſing from the aforeſaid Part in deſcribing $De$, muſt be $\frac{v^r\dot{x}}{bg^{r-1}} \times \frac{b}{g} = \frac{v^{r-1}\dot{x}}{bg^{r-2}}$; from whence, it is manifeſt, by Inſpection, that the other Parts of the Reſiſtance, or Quantities of Motion deſtroyed thereby, will be $\frac{v^{s-1}\dot{x}}{cg^{s-2}}$, $\frac{v^{t-1}\dot{x}}{dg^{t-2}}$, &c. And therefore the whole Velocity deſtroyed by the Medium, in deſcribing $De$, is $\frac{v^{r-1}\dot{x}}{bg^{r-2}} + \frac{v^{s-1}\dot{x}}{cg^{s-2}}$, &c. But, the Time of deſcribing $De$, being to $\frac{b\dot{x}}{g}$ that of deſcribing $Pq$ as $g$ to $v$, will

P be

be repreſented by $\frac{b\dot{x}}{v}$; and therefore it will be, as $b : m :: \frac{b\dot{x}}{v} : \frac{m\dot{x}}{v}$, the Part of Velocity generated or deſtroyed in that Time, by the Force of Gravity, which added to, or taken from, the former Part, ariſing from the Reſiſtance, according as the Body is in its Aſcent or Deſcent, the Sum or Difference $\pm \frac{m\dot{x}}{v} + \frac{v^{r-1}\dot{x}}{bg^{r-2}} + \frac{v^{s-1}\dot{x}}{cg^{s-2}}$, &c. muſt, it is manifeſt, be equal to $(-\dot{v})$ the whole Decrement of Velocity: Hence we have $\dot{x} = \frac{-v\dot{v}}{\pm m + \frac{v^r}{bg^{r-2}} + \frac{v^s}{cg^{s-2}}, \text{ \&c.}}$.

Moreover, becauſe $\dot{T}$, the Time of deſcribing D*e*, is found equal to $\frac{b\dot{x}}{v}$, we have $\frac{\dot{T}v}{b} = \dot{x}$; which being ſubſtituted inſtead thereof in the other Equation, &c. there will come out $\dot{T} = \frac{-b\dot{v}}{\pm m + \frac{v^r}{bg^{r-2}} + \frac{v^s}{cg^{s-2}}, \text{ \&c.}}$. *Q. E. I.*

## COROL. I.

HENCE, when the Reſiſtance is barely in the ſimple Ratio of the Velocity, then *c*, *d*, &c. being infinite, our Equations become $\dot{x} = \frac{-v\dot{v}}{\pm m + \frac{vg}{b}}$, and $\dot{T}$ equal $\frac{-b\dot{v}}{\pm m + \frac{vg}{b}}$: Whence $x = \frac{\overline{g - v} \times b}{g}$, $\pm \frac{mb^2}{g^2}$ into the Hyp. Log. of $\frac{gv \pm mb}{gg \pm mb}$, and $T = \frac{bb}{g}$ into Hyp. Log. of $\frac{gg \pm mb}{gv \pm mb}$.

COROL.

## COROL. II.

BUT, when the Reſiſtance is, as the Square of the Velocity, $r$ being equal to 2, and $c$, $d$, &c. infinite (as before) the Equations will be $\dot{x}$ equal to $\frac{-v\dot{v}}{\mp m + \frac{v^2}{b}}$, and $\dot{T}$ equal to $\frac{-b\dot{v}}{\mp m + \frac{v^2}{b}}$: Hence $x$ is found equal to $\frac{b}{2} \times \text{Log.}\, \frac{g^2 \pm mb}{v^2 \pm mb}$, wherein, if $v$ be taken $= 0$, we ſhall have $\frac{b}{2} \times$ Hypb. Log. $1 + \frac{g^2}{mb}$, for the Height of the whole Aſcent; but, if $v^2 - mb$ be taken $= 0$, we ſhall have $\sqrt{mb}$ equal to the greateſt Velocity the Body can poſſibly acquire by deſcending; laſtly, if $g$ be taken $= 0$, there will be $\frac{b}{2} \times \text{Log.}\, \frac{mb}{mb - vv}$ for the Diſtance gone over when the Body falls from Reſt; therefore, in that Caſe the Log. $\frac{mb}{mb - vv}$ being $= \frac{2x}{b}$, if $n$ be put for the abſolute Number whoſe Hyperbolic Log. is $-\frac{2x}{b}$ we ſhall get $\frac{mb}{mb - vv} = \frac{1}{n}$, and conſequently $v = \overline{mb}|^{\frac{1}{2}} \times \overline{1-n}|^{\frac{1}{2}}$.

Moreover, with reſpect to the Time, becauſe $\dot{T}$ in the Deſcent of the Body is $= \frac{bb\dot{v}}{mb - vv}$ the Time it ſelf will, in this Caſe, be $\frac{b}{2}\sqrt{\frac{b}{m}}$ into the Hyp. Log. $\frac{\overline{mb}|^{\frac{1}{2}} + v}{\overline{mb}|^{\frac{1}{2}} - v} \times \frac{\overline{mb}|^{\frac{1}{2}} - g}{\overline{mb}|^{\frac{1}{2}} + g}$; and therefore, when the Body deſcends from Reſt, is barely $= \frac{b}{2}\sqrt{\frac{b}{m}} \times \text{Log}\, \frac{\overline{mb}|^{\frac{1}{2}} + v}{\overline{mb}|^{\frac{1}{2}} - v}$, wherein, if the above found Value of $v$ be ſubſtituted, it will be $\frac{b}{2}\sqrt{\frac{b}{m}} \times$

Log.

Log. $\frac{1+\overline{1-n}|^{\frac{1}{2}}}{1-\overline{1-n}|^{\frac{1}{2}}}$: But, in the other Cafe, $\dot{T}$ being $= \frac{-bb\dot{v}}{mb+vv}$ T will be equal to $\frac{b}{m}$ drawn into the Difference of the two Circular Arcs, whofe Tangents are $g$ and $v$, and whofe common Radius is $\sqrt{mb}$. And, in like manner, the Values of $x$ and T may be exhibited by the Quadratures, *&c.* of the Conic Sections, in any other Cafe, where the Refiftance is barely as a fimple Power of the Velocity, whofe Exponent is a rational Number, and alfo, in many Cafes, where the Refiftance is in the Ratio of two different Powers, by Help of the laft *Problem* of this *Treatife.*

## C O R O L. III.

IF $m$ be taken $= 0$, or the Body be fuppofed to be affected by a Medium only, and the Refiftance be barely as a fimple Power ($r$) of the Velocity; then $\dot{x}$ becoming equal $-bg^{r-2}v^{1-r}\dot{v}$, and $\dot{T} = -bbg^{r-2}v^{-r}\dot{v}$, $x$, in this Cafe, will therefore be $= \frac{b-bg^{r-2}v^{2-r}}{2-r}$, and T equal $\frac{bbg^{-1}-bbg^{r-2}v^{1-r}}{1-r}$: Where, if $r$ be taken $= 0, 1, 2, 3,$ *&c.* fucceffively, $x$ will be $\frac{b}{2} - \frac{bv^2}{2g^2}$, $b - \frac{bv}{g}$, $b\,\text{Log.}\,\frac{g}{v}$, $-b + \frac{bg}{v}$, *&c.* and T equal to $\frac{bb}{g} \times \overline{\frac{g-v}{g}}$, $\frac{bb}{g}$ Log. $\frac{g}{v}$, $\frac{bb}{g} \times \overline{\frac{g-v}{v}}$, $\frac{bb}{2g} \times \overline{\frac{gg-vv}{vv}}$, *&c.* refpectively; from whence, by exterminating $v$, we have $x = \frac{gT}{b} \times \overline{1-\frac{gT}{2bb}}$, $T = \frac{bb}{g}$ Log. $\frac{b}{b-x}$, $x = b$ Log. $1 + \frac{g}{b} \times \frac{T}{b}$, $T = b \times \frac{bx+\frac{1}{2}xx}{bg}$, *&c.* expreffing the Relation of the Times and Spaces in the faid Cafes, refpectively.

## SCHOLIUM.

IN Fluids void of Tenacity the Refiftance is in the Duplicate Ratio of the Velocity; and it is found, that a Body in fuch Fluids, by moving over a Space, which is to $\frac{8}{3}$ of its Diameter, as the Denfity of the Body, to that of the Medium, with its Velocity uniformly continued, would meet with a Refiftance fufficient to take away its whole Motion: Therefore, if this Space be taken to reprefent the Value of $b$, in *Cor.* II. by Help of the *Theorems* there given, the Velocity, Time, or Space gone over, will be readily obtained. For an Inftance hereof, let a Ball, whofe Diameter is $\frac{1}{3}$ of a Foot, and whofe Denfity is the fame with that of common Rain-Water, be fuppofed to be projected upwards in a Direction perpendicular to the Horizon, with a Velocity fufficient to carry it uniformly over a Space of 300 Feet in one Second of Time; and let the Heighth of the Afcent, the Times of Afcent and Defcent, with the Velocity generated in Falling, be required. Becaufe, the Denfity of Rain-Water, is to that of Air, as 860 to 1, $b$ will, here, be ($\frac{8}{3} \times \frac{1}{3} \times 860$) 764.4 Feet; and fince the Velocity, which a Body would acquire in one Second of Time by freely defcending *in vacuo*, is fufficient to carry it uniformly over a Diftance of 32.2 Feet in that Time, it will be, as the abfolute Gravity, to the fpecifick Gravity; or, as 860, to 859, fo is 32.2, the faid Feet, to $(32.16 =)$ $m$, $b$ being equal to the Time abovementioned: Wherefore, if for $g$, $b$, $b$ and $m$, their refpective Values 300, 1, 764.4, and 32.16, be fubftituted in the aforefaid *Theorems*, we fhall have, firft, $\frac{b}{2} \times$ Hyp. Log. $1 + \frac{g^2}{mb}$ $(= x) = 630$ Feet for the whole Heighth of the Afcent, $2^d$, $\frac{b}{m} \times$ Arch, whofe Tang. is $g$, and Rad $\sqrt{mb}$, $= 5.48$ Seconds,

conds, the whole Time of Aſcent; $3^d$, $n = .21455$; $4^{th}$, $v = 139$, the Diſtance that would be uniformly deſcribed in one Second with the Velocity acquired by falling; laſtly, $\frac{b}{2}\sqrt{\frac{b}{m}} \times$ Log. $\frac{1+\sqrt{1-n}}{1-\sqrt{1-n}} = 6.85$, the Time of Deſcent. But if the ſame Ball be ſuppoſed to move in Water with the ſame given Velocity; then, the ſpecifick Gravity in that Fluid being nothing, the Body may be conſidered as moving by its innate Force only; and, therefore, the Number of Feet gone over, in any Number of Seconds, denoted by T, will (by *Cor.* III.) be [illegible], into the Hyp. Log. of $1 + 337.5$T.

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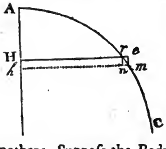
## PROPOSITION II.

*To find the Reſiſtance and Denſity of a Medium, whereby a Body, gravitating uniformly in the Direction of Parallel Lines, is made to deſcribe a given Curve; the Law of Reſiſtance being given, partly as the n Power, partly as the 2 n Power, partly as the 3 n Power, &c. of the Celerity, or as* $aC^n + bC^{2n} + cC^{3n}$, &c. *where* C *denotes the Celerity, and n, a, b, c, &c. any determinate Quantities.*




LET $ArC$ be the propoſed Curve, and AH the Axis thereof, or a Right-line in which the Body gravitates, to which let $rn$ and $em$ be parallel, and $Hr$ and $hm$ perpendicular, $r$ and $m$ being any two Points in the Curve taken indefinitely near to one another: Suppoſe the Body arrived to $r$, with a Velocity in the

the Direction $re$, represented by $v$; let $AH=x$, $Hb\ (rn) = \dot{x}$, $Hr=y$, $nm\ (re)=\dot{y}$, $rm=\dot{z}$, and let D be as the required Density. Then, since the Velocity in the Direction $re$ is $v$, that in the Direction $rn$ will be $\frac{v\dot{x}}{\dot{y}}$; and therefore $\frac{\dot{v}\dot{x}+v\ddot{x}}{\dot{y}}$ will be the Fluxion of the same, or the Increase of Velocity in that Direction during the Time of describing $rm$; wherefore, if from this we take the Part arising from the Resistance of the Medium, which is $\frac{\dot{v}\dot{x}}{\dot{y}}$ (because it is to $\dot{v}$, the Alteration of Velocity in the Direction $re$, as $\dot{x}$ to $\dot{y}$) there will remain $\frac{v\ddot{x}}{\dot{y}}$ for the other Part arising from the Force of Gravity, in the same Time and Direction; therefore, the Resistance in the said Direction, is to the Force of Gravity, as $-\frac{\dot{v}\dot{x}}{\dot{y}}$ to $\frac{v\ddot{x}}{\dot{y}}$, or, as $-\frac{\dot{v}\dot{x}}{v\ddot{x}}$ to $1$; and, consequently, the absolute Resistance, in the Direction $rm$, to the Force of Gravity, as $\frac{-\dot{v}\dot{z}}{v\ddot{x}}$ to $1$: But $\left(\frac{v\ddot{x}}{\dot{y}}\right)$ the Part of Velocity, arising from Gravity, being as the Time $\frac{\dot{z}}{v}$ of describing $rm$, may be expressed thereby; whence we have $\frac{v\ddot{x}}{\dot{y}}=\frac{\dot{z}}{v}$, or $v^2\ddot{x}=\dot{y}\dot{z}$; and therefore in Fluxions $2v\dot{v}\ddot{x} + v^2\dddot{x}=0$, or $-\frac{\dot{v}}{v}=\frac{\dddot{x}}{2\ddot{x}}$, which substituted in the foregoing Proportion $-\frac{\dot{v}\dot{z}}{v\ddot{x}} : 1$ gives $\frac{\dot{z}\dddot{x}}{2\ddot{x}\ddot{x}}$ to $1$ for the Ratio of the Resistance to the Gravity. Moreover, because the

the abſolute Velocity is $\frac{v\dot z}{\dot y}$, the Reſiſtance, by Suppoſition, will be as D into $a\times\overline{\frac{v\dot z}{\dot y}}\Big|^{n}+b\times\overline{\frac{v\dot z}{\dot y}}\Big|^{2n}$ &c. or, becauſe $\frac{v\dot z}{\dot y}$ is $=\frac{\dot z}{\ddot x|^{\frac{1}{2}}}$, as D into $a\times\frac{\dot z}{\ddot x^{\frac{n}{2}}}+b\times\frac{\dot z^{2n}}{\ddot x^{n}}$, &c. which Quantity muſt therefore be as $\frac{\dot z\dddot x}{\dot x\ddot x}$, and conſequently D as

$$\frac{\dddot x}{a\dot z^{n-1}\times\ddot x^{\frac{4-n}{2}}+b\dot z^{2n-1}\ddot x^{\frac{4-2n}{2}}+c\dot z^{3n-1}\ddot x^{\frac{4-3n}{2}}\ \&c.}\quad Q.E.I.$$

## COROLLARY.

HENCE, if the Law of Reſiſtance be only as a ſingle Power ($n$) of the Velocity; then, by taking $b$, $c$, $d$, &c. each $=0$, and $a=1$, we have $\frac{\dddot x}{\dot z^{n-1}\times\ddot x^{\frac{4-n}{2}}}$ for the Denſity in that Caſe; which, therefore, when $n$ is $=2$, or the Reſiſtance directly as the Square of the Velocity, will be barely as $\frac{\dddot x}{\dot z\ddot x}$.

## EXAMPLE I

LET it be required to find the Denſity of a Medium, wherein a Body moving, ſhall deſcribe the common Parabola. Here $x$ being $=\frac{y^2}{p}$, we have $\dot x$ equal

$\frac{2y\dot y}{p}$

$\frac{2\dot{y}y}{p}$, $\dot{x}=\frac{2\dot{y}\dot{y}}{p}$; and $\dddot{x}=0$; and therefore $D=0$; which shews, that a Body, to describe this Curve, must move in Spaces entirely void of Resistance.

## EXAMPLE II.

TO find the Density, &c. when the Curve is a Circle, and the Resistance as the Square of the Celerity.

Because $x$, in this Case, is $=a-\sqrt{aa-yy}$, there will be $\dot{x}=\frac{y\dot{y}}{\sqrt{aa-yy}}$, $\dot{z}=\frac{a\dot{y}}{\sqrt{aa-yy}}$, $\ddot{x}=\frac{aa\dot{y}\dot{y}}{\overline{aa-yy}|^{\frac{3}{2}}}$, and $\dddot{x}=\frac{3a^2y\dot{y}^3}{\overline{aa-yy}|^{\frac{5}{2}}}$; therefore the Density $\left(\frac{\dddot{x}}{\dot{z}\ddot{x}}\right)$ will here be, as $\frac{3y}{a\sqrt{aa-yy}}$, or, as the Tangent of the Distance from the highest Point directly; and the Resistance will be to Gravity, as $3y$ to $2a$, or, as 3 Times the Sine of the same Distance to twice the Radius.

## SCHOLIUM.

IF the Density of the Medium be given, the Curve it self may be determined by the Construction of the foregoing fluxional Equation: So, in case of an uniform Density, and a Resistance, as the Square of the Velocity, where we have $D=\frac{\dddot{x}}{\dot{z}\ddot{x}}$, or $D\ddot{x}\sqrt{\dot{y}\dot{y}+\dot{x}\dot{x}}=\dddot{x}$; $x$ will be found $=\frac{y^2}{2p}+\frac{Dy^3}{3p}+\frac{D^2y^4}{12p}$, &c. And when $D$ is constant, and the Resistance barely as the Velocity, it

will be $x = -\frac{4}{DD} \times \text{Log.}\ \overline{1 - \frac{Dy}{2p^{\frac{1}{2}}}}, - \frac{4v}{2p|^{\frac{1}{2}} \times D}$; $p$, in either Case, being twice the Radius of Curvature, or the Parameter at the Vertex; both which, and the true Value of D, may be easily computed from the Velocity at A, and the given Density of the Medium.

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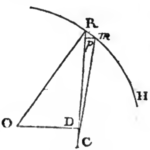
## PROPOSITION III.

*The Centripetal Force being given, and the Law of Resistance, as any Power (n) of the Velocity; to find the Density of a Medium in each Part thereof, whereby a Body may describe a given Spiral about the Centre of Force.*




LET R$m$H be the given Spiral, R and $m$ two Points therein as near as may be to each other, and C the Centre of Force; and let R O be the Radius of Curvature at R; making O D and $mp$ perpendicular to R C, and calling R D, $s$; R C, $x$; R $p$, $\dot{x}$; R $m$, $\dot{z}$; the Centripetal Force, C; and the Velocity, $v$. Forasmuch, as R O is to R D, as the absolute Centripetal Force at R, to that, which tending to the Centre O, would be sufficient to retain the Body in the Circle, whose Radius is R O, and, because the Centripetal Forces in Circles, the Velocities being the same, are inversely as the Radii; R D, it is manifest, is the Radius of the Circle which might be described with the Velocity and Centripetal Force at R: Therefore, since the Centripetal Force,

in

in Circles, is known to be such, as is sufficient to generate or destroy all the Velocity of the moving Body, in the Time it is uniformly describing a Distance equal to the Semi-diameter of its Orbit, we have, $s$ (RD) : $\dot{z}$ (R$m$) :: $v$ : $\left(\frac{v\dot{z}}{s}\right)$ the Velocity which the Centripetal Force would generate in another Body freely descending from Rest at R, in the Time the former is describing R$m$; wherefore, by the Resolution of Forces, it will be $\dot{z}$ (R$m$) : $\dot{x}$ (R$p$) :: $\frac{v\dot{z}}{s} : \frac{v\dot{x}}{s}$, the Velocity generated in the same Time, by the Body describing R$m$; which, therefore, added to $\dot{v}$ the Excess of the Velocity at R above that at $m$, the Sum $\frac{v\dot{x}}{s} + \dot{v}$ will, it is manifest, express the Velocity taken away by the Medium in that same Time: But the Velocities generated or destroyed in equal Bodies, in equal Times, are as the Forces by which they are generated or destroyed; and, therefore, it will be as $\frac{v\dot{x}}{s} + \dot{v} : \frac{v\dot{z}}{s}$, or as $\frac{\dot{x}}{\dot{z}} + \frac{s}{\dot{z}} \times \frac{\dot{v}}{v}$ to 1, so is the Resistance to the Centripetal Force. But, the Velocities in Circles being in the subduplicate Ratio of the Radii and Centripetal Forces conjunctly, $v$ will be as $\sqrt{sC}$, and consequently $\frac{\dot{v}}{v} = \frac{\dot{s}}{2s} + \frac{\dot{C}}{2C}$; whence, by Substitution, it will be as $\frac{2\dot{x}+\dot{s}}{2\dot{z}} + \frac{s\dot{C}}{2C\dot{z}} : 1$, or, as $\overline{2\dot{x}+\dot{s}} \times \frac{C}{2\dot{z}} + \frac{s\dot{C}}{2\dot{z}} : C$, so is the Resistance to the Centripetal Force; but C is the Centripetal Force, and therefore $\overline{2\dot{x}+\dot{s}} \times \frac{C}{2\dot{z}} + \frac{s\dot{C}}{2\dot{z}}$ is the required Resistance; which being divided by $(\overline{sC}|^{\frac{n}{2}})$ the

the $n$ Power of the Velocity, becauſe the Reſiſtance is in the Ratio thereof, and the Denſity of the Medium conjunctly, the Quotient $\frac{2\dot{x}+\dot{s}}{2 s^{\frac{n}{2}} \dot{z} C^{\frac{n-2}{2}}} + \frac{\dot{C}}{2 C^{\frac{n}{2}} \dot{z} s^{\frac{n-2}{2}}}$ will, it is manifeſt, be as the Denſity of the Medium. *Q. E. I.*

## EXAMPLE.

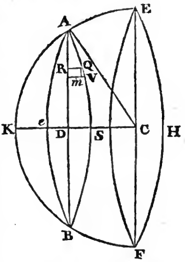
LET the Reſiſtance be in the duplicate Ratio of the Velocity, the Centripetal Force as ſome Power, $m$, of the Diſtance, and the Curve propoſed the Logarithmic Spiral; and, Radius being $r$, let $c$ be the Co-ſine of the common Angle, which all the Ordinates make with the Spiral: Then $s$, by the Nature of the Curve, being $= x$ ($=$ C R) $\frac{2\dot{x}+\dot{s}}{2\dot{z}}$ will be $= \frac{3\dot{x}}{2\dot{z}} = \frac{3c}{r}$, and therefore $\frac{2\dot{x}+\dot{s}}{2\dot{z}} + \frac{s\dot{C}}{2C\dot{z}} = \frac{3c}{2r} + \frac{m\dot{x}}{2\dot{z}} = \frac{c}{r} \times \frac{m+3}{2}$; hence we have, as $\frac{m+3}{2} : \frac{r}{c}$, ſo is the Reſiſtance to the Centripetal Force. And the Denſity of the Medium will be as $\frac{c \times \overline{m+3}}{2rx}$; that is, when $c$ and $m$ are given, reciprocally as the Diſtances from the Centre of Force: But when $m$ is $-3$, then $\frac{c \times \overline{m+3}}{2rx}$ becoming $= 0$, it appears, that the Body in this Caſe muſt move in Spaces entirely void of Reſiſtance to deſcribe the propoſed Spiral: And, therefore, the Law of Centripetal Force being more than the Cube of the Diſtance inverſely, the Deſcription of this Curve will, it is manifeſt, be impoſſible from any reſiſting Force whatſoever.

Of

## Of the Motion and Refiftance of Pendulous Bodies in a Medium.

### PROPOSITION I.

*Suppofing two equal Pendulums, whofe Bobs are in Form of the Segments of Spheres, to be moving with equal Velocities in a refifting Medium, and the Thicknefs of each Bob with the Diameter of the Sphere from which it is formed, to be given; To find the Ratio of their Refiftances.*




LET AKBA be one Side, or Half, of one of the propofed Bobs, and EAKBFCE Half the whole Sphere whereof it is a Segment; and let the faid Segment be conceived to be divided into an indefinite Number of indefinitely fmall *Laminæ*, by circular Planes perpendicular to the Axis KC, and equi-diftant from each other; and let A*e*BSA be one of thofe *Laminæ*, included between any two adjacent Planes, and $n$ be the Thicknefs thereof, or the common Diftance of the faid Planes; calling AC, $a$; AD, $c$; any Ordinate RQ, $y$; $mv$, $\dot{y}$; and KD, $x$. Then, by the Property of the Circle, we have $DR = \sqrt{cc - yy}$; which, Radius being $a$, is the Sine of the Angle that the Surface at Q makes

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with $m$ Q, the Direction of its Motion, or the Incidence of the Particles it ſtrikes againſt: Therefore, ſince the reſiſting Force of the Medium, on any Surface the Velocity being the ſame, is as the Number of Particles falling thereon, and the Square of the Sine of their common Incidence conjunctly, the whole Reſiſtance of that Part of the propoſed Surface, repreſented by Q$v$, will be as $\frac{cc-yy}{aa} \times n\dot{y}$; becauſe $n\dot{y}$ is evidently as the Number of Particles: Hence, by taking the Fluent, we have $\frac{ny}{aa} \times \overline{cc - \frac{1}{3}y^2}$ for the Reſiſtance of A Q; the Double whereof, $\frac{4c^3n}{3aa}$, when $y$ becomes equal $c$, will conſequently be the whole Reſiſtance of the ſaid *Lamina*: Which Reſiſtance, if the Axis KD ($x$) be, now, ſuppoſed to flow, and $\dot{x}$ be put inſtead of $n$, will, it is manifeſt, be the Fluxion of the Reſiſtance of the propoſed Segment AKBD: But $x$ being $= a - \sqrt{aa - cc}$ by the Property of the Circle $\dot{x}$ will $\therefore$ be $= \frac{c\dot{c}}{\sqrt{aa-cc}}$, and conſequently the above-ſaid Fluxion equal to $\frac{4c^4\dot{c}}{3aa\sqrt{aa-cc}}$; whoſe Fluent will be $\frac{AK \times CK}{2} - \frac{AD \times DC}{2} \times \overline{1 + \frac{2AD^2}{3CK^2}}$, and therefore the Double thereof as the whole Reſiſtance of the given Pendulum. *Q. E. I.*

### COROL. I.

HENCE it appears, that the Reſiſtance of the whole generating Sphere will be expreſs'd by EK × KC, or the Area of the Semi-circle EKFE; and therefore is to that of its circumſcribing Cylinder, moving in the Direction of its Axis, exactly, as 1 to 2.

COROL.

## COROL. II.

IF the Resistance, as above found, be divided by 3.14159, &c. $\times KD^2 \times \frac{6KC-2KD}{3}$, the solid Content, or Quantity of Matter in the Pendulum, the Quotient will, it is manifest, be as the Retardation of its Velocity arising from that Resistance; and this, if $b$ be put for the Axis or Thickness of the Bob, and $d$ its greatest Diameter, will be equal to $\frac{3bbd}{\overline{bb+dd}^2}$ very nearly.

## COROL. III.

WHEREFORE, if $b$ be taken $= d$, we shall have $\frac{3}{4d}$ for the Retardation of the Globe, whose Diameter is $d$; and therefore the Retardation of the Pendulum to that of its circumscribing Sphere will be as $\frac{3bbd}{\overline{bb+dd}^2}$ to $\frac{3}{4d}$, or as $\overline{2bd}^2 : \overline{bb+dd}^2$ nearly.

*Note*, If the Bobs of Pendulums be in other Forms than those of Segments of Spheres, the Resistance will be readily had as above; since it is evident, that A C ($a$) being taken for the Normal of the generating Curve K A, the Resistance of the *Lamina* A$e$BSA will be $\frac{4c^3n}{3aa}$ as is there found, let that Curve be what it will.

## LEMMA.

*The Resistance of a Body in a Medium, is to the Force of Gravity, as twice the Space thro' which the Body must freely fall by that Gravity to acquire the given Velocity, to the Space*

*Space over which it might move with that Velocity in the Time wherein the said Resistance, uniformly continued, would take away the Body's whole Motion.*

For the Velocity acquired by freely descending from Rest, thro' any Space, is known to be generated in the Time that the Body, with the Velocity so acquired, would move uniformly over double that Space: But the Forces, by which the same Motion would be uniformly generated or destroyed, are inversely as the Times in which it might be generated or destroyed; and therefore inversely, as the Distances described with the same Velocity in those Times.

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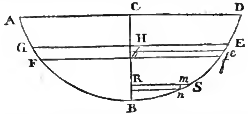
## PROPOSITION II.

*Supposing that a heavy pendulous Body, oscillating in a Cycloid, is resisted by an uniform Force, and at the same time by a rare and similar Medium, in the duplicate Ratio of the Velocity; To find the Excess of the Arc, described in the whole Descent above the Arc, described in the subsequent Ascent, and the Time of one entire Oscillation.*

LET ABD be the whole Cycloid, BC its Axis, EB the Arc described in the Descent, and BF that de-




scribed in the subsequent Ascent; draw GHE, F*f*, &c. parallel to AD, let S be any Place of the Body, and *b* R, the

the Diſtance thro' which it muſt freely deſcend in *vacuo*, by a Force equal to its ſpecifick Gravity in the given Medium, to acquire the ſame Velocity as it has in that Place. Suppoſe that Part of the Reſiſtance, which is uniform, to be to the Force of Gravity, as $m$ to $1$; and let $d$ be the Space over which the Body muſt move with its Velocity uniformly continued, to meet with a Reſiſtance from the other Part, ſufficient to take away its whole Motion; or, which is to the ſame Effect, let $d$ be to the indefinitely ſmall Arc $Sn$, as the whole Motion of the Body at S, to that deſtroyed by the Medium in moving thro' $Sn$; calling BD, $b$; BE, $a$; R$b$, $z$; ES, $x$; $Sn$, $\dot{x}$. Now, the Force of Gravity being repreſented by $1$, that Part of it whereby the Body is accelerated, at the Point S, is $\frac{a-x}{b}$ $\left(=\frac{BS}{BD}\right)$ : And, by the preceding *Lemma*, we have, as $d : 2z :: 1 : \frac{2z}{d}$ for the latter Part of the Reſiſtance, or that in the duplicate Ratio of the Velocity; which being added to $m$, the former Part, and the whole taken from $\frac{a-x}{b}$, gives $\frac{a-x}{b} - m - \frac{2z}{d}$ for the whole Force whereby the Body is accelerated at the ſaid Point: Therefore the Velocity, there, being known to be, as $\sqrt{2z}$, that generated in $\frac{\dot{x}}{\sqrt{2z}}$ the Time of deſcribing $Sn$ will be defined by $\overline{\frac{a-x}{b} - m - \frac{2z}{d}} \times \frac{\dot{x}}{\sqrt{2z}}$; which muſt therefore be equal to, $\frac{\dot{z}}{\sqrt{2z}}$, the Fluxion or Increaſe of, $\sqrt{2z}$, the aforeſaid Velocity: Hence, we have $\frac{a-x}{b} - m - \frac{2z}{d} = \frac{\dot{z}}{\dot{x}}$; which, by writing $e$ inſtead

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of $a - mb$, &c. becomes $-\frac{e}{b} + \frac{x}{b} + \frac{2z}{d} + \frac{\dot z}{\dot x}$ equal o; from whence, by solving the Equation, $z$ is found $= \frac{ex}{b} - \overline{\frac{d+2e}{2bd} \times x^2 - \frac{2x^3}{3d} + \frac{4x^4}{3\cdot 4d^2} - \frac{8x^5}{3\cdot 4\cdot 5d^3}}$, &c. Or, if $p$ be put for (0.367878) the Number, whose hyperbolical Logarithm is $-1$, $= 1 - p^{\frac{2x}{d}} \times \frac{\frac{1}{2}dd + de}{2b} - \frac{dx}{2b}$. But when the Body arrives at F, the Hight of its Ascent, $z$, or its Equal $\frac{ex}{b} - \overline{\frac{d+2e}{2bd} \times x^2 - \frac{2x^3}{3d} + \frac{4x^4}{3\cdot 4d^2}}$, &c. becomes equal o; which Equation solved, gives $x = 2e - \frac{4e^2}{3d} + \frac{16e^3}{9d^2}$, &c. $=$ EBF; therefore FG is $= 2a - 2e + \frac{4e^2}{3d} - \frac{16e^3}{9dd}$, &c. $= 2mb + \frac{4e^2}{3d} - \frac{16e^3}{9dd}$, &c. (by resuming $2mb$ instead of its Equal $2a - 2e$;) which, because $m$ and $\frac{e}{d}$ are supposed very small, will be $2mb + \frac{4a^2}{3d}$ very nearly. *Q. E. I.*

Moreover, since the Time of describing the least possible Distance S$n$, is as $\frac{\dot x}{\sqrt{2z}}$, by substituting therein the Value of $z$, as above found, we shall have

$$\frac{\dot x}{\overline{\frac{2ex}{b} - \frac{d+2e}{bd} \times x^2 - \frac{2x^3}{3d} + \frac{4x^4}{3\cdot 4\cdot d^2}, \text{ \&c.}}|^{\frac{1}{2}}} = \frac{b^{\frac{1}{2}}}{\overline{1 + \frac{2e}{d}}|^{\frac{1}{2}} \times x^{\frac{1}{2}}}$$

$$\times \frac{\dot x}{\overline{\frac{2e}{1+\frac{2e}{d}} - x + \frac{2x^2}{3d} - \frac{4x^3}{3\cdot 4d^2}, \text{ \&c.}}|^{\frac{1}{2}}}$$ for the Fluxion of the required Time: But, because $\frac{2e}{1+\frac{2e}{d}} - x + \frac{2x^2}{3d}$, &c. the Square of the Divisor of the latter Part or Factor thereof, when $x$ becomes

becomes $= 2e - \frac{4ee}{3d} + \frac{16e^3}{9d^2}$, &c. appears, from above, to be equal to Nothing, if $r$ be put to denote the Value of $2e - \frac{4ee}{3d} + \frac{16e^3}{9dd}$, &c. or the Root of the Equation, it is manifest that $\frac{2e}{1+\frac{2e}{d}} - r + \frac{2r^2}{3d} - \frac{4r^3}{3.4dd}$ will also be equal o, and consequently $r - \frac{2r^2}{3d} + \frac{4r^3}{3.4dd}$, &c. $= \frac{2e}{1+\frac{2e}{d}}$; which being substituted instead thereof, our said Fluxion will become $\frac{b^{\frac{1}{2}}}{\overline{1+\frac{2e}{d}}|^{\frac{1}{2}} \times x^{\frac{1}{2}}} \times \frac{\dot{x}}{\overline{r - \frac{2r^2}{3d} + \frac{4r^3}{3.4dd}, \&c. - x + \frac{2x^2}{3d}, \&c.}|^{\frac{1}{2}}}$

$= \frac{b^{\frac{1}{2}}}{\overline{1+\frac{2e}{d}}|^{\frac{1}{2}} \times x^{\frac{1}{2}}} \times \frac{\dot{x}}{\overline{r - x \text{ in } 1 - 2 \times \frac{r+x}{3d} + 4 \times \frac{rr+rx+xx}{3.4dd}, \&c.}|^{\frac{1}{2}}} =$

$\frac{b^{\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}}{\overline{1+\frac{2e}{d}}|^{\frac{1}{2}} \times \overline{r-x}|^{\frac{1}{2}}}$ into $\overline{1 - 2 \times \frac{r+x}{3d} + 4 \times \frac{rr+rx+xx}{3.4dd}, \&c.}|^{-\frac{1}{2}}$,

and lastly, by converting $\overline{1 - 2 \times \frac{r+x}{3d}, \&c.}|^{-\frac{1}{2}}$ into a rational Series, equal to $\frac{b^{\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}}{\overline{1+\frac{2e}{d}}|^{\frac{1}{2}} \times \overline{r-x}|^{\frac{1}{2}}}$ into $\overline{1 + \frac{r+x}{3d} + \frac{rx}{6dd}}$ &c. Now, the Fluent of $\frac{x^{-\frac{1}{2}} \dot{x}}{\overline{r-x}|^{\frac{1}{2}}}$ $\left(= \frac{\dot{x}}{\sqrt{rx - xx}}\right)$ when $x$ is $= r$, is known to be equal to the Periphery of the Circle whose Diameter is Unity; wherefore, if that Periphery be put equal $p$, the required Fluent of our given Expression $\frac{b^{\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}}{\overline{1+\frac{2e}{d}}|^{\frac{1}{2}} \times \overline{r-x}|^{\frac{1}{2}}} \times \overline{1 + \frac{r+x}{3d} + \frac{rx}{6dd}}$, &c. will then appear

to

to be $\frac{b^{\frac{1}{2}}p}{\overline{1+\frac{2e}{d}}|^{\frac{1}{2}}} \times \overline{1 + \frac{r + \frac{1}{2}r}{3d} + \frac{rr}{12dd}}$, &c. from *p.* 118. of my Book of *Fluxions*; which, by reſtoring the known Value of $r$, will become $pb^{\frac{1}{2}} \times \overline{1 + \frac{2e}{d}}|^{-\frac{1}{2}} \times \overline{1 + \frac{e}{d} - \frac{e^2}{3dd}}$, &c. $= pb^{\frac{1}{2}} \times \overline{1 - \frac{e}{d} + \frac{3e^2}{2dd} - \frac{5e^3}{2d^3}}$, &c. $\times \overline{1 + \frac{e}{d} - \frac{e^2}{3dd} + \frac{4e^3}{9d^3}}$ &c. $= pb^{\frac{1}{2}} \times \overline{1 + \frac{e^2}{6dd} - \frac{2e^3}{9d^3}}$, &c. and this is, it is manifeſt, as the Time of one entire Oſcillation. *Q. E. I.*

## COROL. I.

WHEN $d$ is infinite, then FG becoming barely equal $2mb$, it will be as $2b : FG :: 1 : m$: Hence it appears, that the Exceſs of the Arc deſcribed in the whole Deſcent above that deſcribed in the ſubſequent Aſcent, when the Reſiſtance is uniform, is to twice the Length of the Pendulum, or DBA, as the reſiſting Force, to the Force of Gravity.

## COROL. II.

BUT, when $m$ is $= 0$, or the Reſiſtance barely in the duplicate Ratio of the Velocity; the ſaid Exceſs will be in the duplicate Ratio of the Velocity, or Arc deſcribed nearly.

## COROL. III.

IF $m$ be conſidered as negative, or the Pendulum, inſtead of being reſiſted by an uniform Force, be accelerated thereby, ſo as to continue its Vibrations in the ſame given

Arc;

Arc; then, since $2mb + \frac{4aa}{3d}$ (= F G) becomes = o, $-m$ will be $= \frac{2aa}{3bd}$ : And, therefore, it is manifest, that the Force, which acting uniformly on the given Body, is sufficient to counter-ballance a Resistance in the duplicate Ratio of the Velocity, or to keep it vibrating in the same given Arc, must be to the Weight of the Pendulum, as $2aa : 3bd$ nearly: And, therefore, the Arcs, which a given Pendulum so actuated, will continue to describe, by different Forces, will be nearly as the Square Roots of those Forces.

## COROL. IV.

WHEN both $m$ and $\frac{a}{d}$ are equal to nothing, that is, when the Oscillations are performed without Resistance, the Time of Vibration will be barely $pb^{\frac{1}{2}}$; which is to the Time, wherein a Body freely descending from Rest, would fall thro' C B, Half the Length of the Pendulum, as the Circumference of a Circle, to its Diameter.

## COROL. V.

MOREOVER, when only $\frac{a}{d}$ is = o, or the Resistance uniform, the Vibrations will, also, be Isochronal, and performed in the very same Time as if the Pendulum was not at all resisted.

## COROL. VI.

BUT if $m$ be equal to nothing, or the Pendulum be resisted, only, in the duplicate Ratio of the Velocity; the Time of Oscillation will then be $pb^{\frac{1}{2}} \times$

$\overline{1+\frac{a^2}{6dd}-\frac{2a^3}{9d^3}}$, &c. Therefore, the Excefs of the Time of one whole Vibration, in a Medium refifting in the duplicate Ratio of the Velocity, above the Time of Vibration in the leaft Arc poffible, is to the Time of Vibration in this Arc, as $\frac{aa}{6dd}-\frac{2a^3}{9ddd}$, &c. to Unity; or, becaufe $\frac{a}{d}$ is very fmall, as $\frac{aa}{6dd}$, to 1 very nearly: Hence it fhould follow, that the faid Excefs, is in the duplicate Ratio of the Arcs very nearly, I fay *fhould follow*, becaufe I know very well, that Sir *Ifaac Newton*, Princip. Prop. 27. B. II. makes it to be, nearly, in the fimple Ratio of the Arcs: This I confefs had made me more than a little fufpect, that I might have here fallen into an Error; and yet upon re-examining the Procefs with more than ordinary Attention, I have not been able to difcover any Miftake therein committed; but, if any fuch fhould occur to my Readers, I fhall readily acknowledge my felf obliged for the Difcovery.

## SCHOLIUM.

IF inftead of a Cycloid, the Ofcillations be performed in a Circle, the above Conclufions will ftill hold, provided the Arc defcribed be but fmall; excepting thofe that relate to the Time of Vibration, which is fhortned or prolonged, independent of the Refiftance, from the particular Nature of the Curve, according as a fmaller or greater Arc is defcribed. But, if to the Time $pb^{\frac{1}{2}}\times\overline{1+\frac{a^2}{6d^2}-\frac{2a^3}{9d^3}}$, &c. found as above, be added the Excefs of the Time of Vibration in the Arch $a$, of a Circle whofe Radius is $b$, above the Time, in the leaft Arc poffible, which, by *p.* 140. of my Book of

*Fluxions,*

*Fluxions*, is $pb^{\frac{1}{2}} \times \overline{\frac{aa}{10bb}}$, &c. we shall then have $pb^{\frac{1}{2}} \times \overline{1 + \frac{a^2}{10bb} + \frac{a^2}{6d^2}}$, &c. for the Time of Oscillation in the Arc $a$ of that Circle nearly. Hence it will not be difficult to determine, how much the Times of Vibration, in small Arcs of Circles, are increased or decreased from the different Weights of the Atmosphere. For, if the Force by which the Pendulum is kept in Motion, be always the same, the Arc described, by *Cor.* III. will be as $\sqrt{bd}$, that is, in the subduplicate Ratio, inversely, of the Density of the Medium, or Height of the Barometer: Therefore, if $b$ be put for the Height of the Barometer, at the Time when a given Arc $c$ is described, the Length of the Vibration corresponding to ($y$) any other Height thereof, will be $\frac{cb^{\frac{1}{2}}}{y^{\frac{1}{2}}}$; this therefore being substituted instead of $a$, and $\frac{db}{y}$ instead of $d$, in the above Expression, gives $pb^{\frac{1}{2}} \times \overline{1 + \frac{bc^2}{10yb^2} + \frac{yc^2}{6bdd}}$, &c. for the Time of Vibration corresponding to this last Height; which, when $y = b$, is $pb^{\frac{1}{2}} \times \overline{1 + \frac{c^2}{10bb}}$, &c. therefore the Difference of the Times of Vibration answering to the two Heights of the Barometer $b$ and $y$, if $n$ be put equal to the Difference of those Heights, will be $\frac{pb^{\frac{1}{2}}n}{y} \times \frac{c^2}{16bb} - \frac{yc^2}{6bad}$, &c. nearly, excepting by so much as it is varied thro' the different specific Gravities of the Pendulum, &c. in a rarer or denser Atmosphere; which is easy to be computed. However, after all, it is not to be supposed, that the Alteration in the Time of Vibration, above specified, will happen

happen immediately upon the Rife or Fall of the Mercury; becaufe the Pendulum, thro' its *vis inertiæ* will be fome Time before it can be brought to perform its Vibrations, either in a greater or fmaller Arc: And, indeed, the Alterations, both in the Time and Arc defcribed, from the above Caufe, are fo fmall, when compared with thofe arifing from Friction and Expanfion, as fcarcely to come under the niceft Obfervation.

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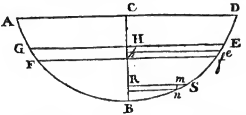
## PROPOSITION III.

*Suppofing that a heavy Body, ofcillating in a Cycloid, is refifted by a rare and fimilar Medium in the Ratio of a given Power of (n) of the Velocity; to find the Excefs of the Arc defcribed in the whole Defcent above that defcribed in the fubfequent Afcent, and the Number of Ofcillations that will be performed before any other given Arc is defcribed, or the Pendulum has loft a given Part of its Motion.*

LET ABD be the whole Cycloid, BC its Axis, EB the given Arc defcribed in the firft Defcent, BF that defcribed in the fubfequent Afcent, and FG the required




Difference of thofe Arcs; and, fuppofing the Body to be arrived at any Point S, let its Velocity there be the fame as it

it would acquire in freely defcending from Reft thro' the Arc $eS$ by an uniform Gravity equal to its fpecifick Gravity in the given Medium; and let $d$ be to the Length of the Arc BD, as the faid Gravity to the Refiftance, which the Body would fuffer with the Velocity that it might acquire, from that Gravity, by freely falling thro' CB: Draw EHG, SR, &c. parallel to AD, and let $b$, $a$, $A^{\frac{1}{2}}$, and $z$, ftand for BD, BE, B$e$, and BS refpectively. Therefore, the Velocity acquired *in vacuo*, being in the fubduplicate Ratio of the Diftance perpendicularly defcended, $\overline{Rb}|^{\frac{1}{2}}$, or its Equal $\sqrt{A - zz}$ (from the Property of the Curve) will be as the Velocity at S; and therefore $\frac{-\frac{1}{2}\dot{A} + z\dot{z}}{\sqrt{A - zz}}$ the Fluxion thereof, as the Increafe of that Velocity: But, this Increafe depends upon two Caufes; the one, the Force of Gravity, and the other, the Refiftance of the Medium. If the Medium did not refift, A would be conftant, and therefore the Increafe of the Velocity barely as $\frac{z\dot{z}}{\sqrt{A - zz}}$; wherefore, the other Part, arifing from the Refiftance, muft be as $\frac{-\frac{1}{2}\dot{A}}{\sqrt{A - zz}}$, and, confequently, the refifting Force of the Medium, to that Part of the Gravity by which the Body is accelerated, as $\frac{\frac{1}{2}\dot{A}}{\sqrt{A - zz}} : \frac{z\dot{z}}{\sqrt{A - zz}}$, or, as $\dot{A}$ to $2z\dot{z}$: But this Part of the Gravity being to the whole, as $z$ to $b$, by the Property of the Curve; the Refiftance will, it is manifeft, be to the fpecific Gravity, as $\dot{A}$ to $2b\dot{z}$. Moreover, becaufe the Velocity, at the fame Point S, is to the Velocity which would be acquired by freely defcending (as above) along BC, as $\sqrt{A - zz} : b$; the Refiftance, with the former of thofe Velocities, will be to the Refiftance with the

 latter,

latter, as $\overline{A-zz}|^{\frac{n}{2}}$ to $b^n$, or, as $\frac{\overline{A-zz}|^{\frac{n}{2}}}{b^{n-1}}$ to $b$; and, consequently, the Resistance at S, to the Force of Gravity, as $\frac{\overline{A-zz}|^{\frac{n}{2}}}{b^{n-1}}$ to $d$: Wherefore, it will be as $\frac{\overline{A-zz}|^{\frac{n}{2}}}{b^{n-1}} : d ::$ $\dot{A} : 2b\dot{z}$; whence $\frac{\dot{A}}{2} = \frac{\dot{z}\times\overline{A-zz}|^{n}}{db^{n-2}}$, from which Equation A may be determined by the known Methods of infinite Series, *&c.* be the Medium what it will: But, in a very rare one, such as is supposed in the *Proposition*, the Thing may be, otherwise, much more easily effected. For, then E*e*, being, at its greatest, exceeding small, $aa-zz$, may, without sensible Error, be substituted in our Equation for $A-zz$; which done, we have $\frac{\dot{A}}{2}$ equal $\dot{z}\times\frac{\overline{aa-zz}|^{\frac{n}{2}}}{db^{n-2}}$: And then the Fluent of the latter Part thereof, when $z$ is equal $a$, and $n$ an even Number, will be found to come out $\frac{a^{n+1}}{db^{n-2}}$ into $\frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}\times\frac{8}{9}$, *&c.* to $\frac{n}{2}$ Factors; and when $z$ equal $a$, and $n$ an odd Number, equal to $\frac{3.14159.\ \&c.\times a^{n+1}}{2db^{n-2}}$ in $\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\times\frac{7}{8}$, *&c.* to $\frac{n+1}{2}$ Factors; from which two Expressions, the Fluent of the said Part, in any intermediate Case, where $n$ is a Fraction, may, by Interpolation, be nearly obtained: But, $\frac{A}{2}$ the Fluent of the contrary Side, when S coincides with B, is $=\frac{Be^2}{2}$, and when S and $e$ coincide with E, equal

$\frac{BE^2}{2}$

$\frac{BE^2}{2}$; wherefore, if $s$ be put for the Uncia of the Fluent, above found, we ſhall have $\frac{sa^{n+1}}{db^{n-2}}$ equal $\frac{BE^2 - Be^2}{2}$ equal $\frac{BE + Be \times BE - Be}{2} = \frac{\overline{BE + Be} \times Ee}{2}$; whence, becauſe $Be$ is nearly equal to B E, we get $Ee = \frac{sa^n}{db^{n-2}}$, that is in the aforeſaid Circumſtance, when S coincides with B; therefore $\frac{2sa^n}{db^{n-2}}$, the Double thereof, will, it is evident, be equal to F G very nearly. *Q. E. I.*

Let $x$, now, be the Number of Vibrations performed before the Pendulum is brought to deſcribe, in its whole Deſcent, any ſmaller Arc (E): Then, ſince ($\dot{E}$) the Decrement of this Arc, in one entire Oſcillation, or while $x$ is increaſed by 1, or $\dot{x}$, is found to be $= \frac{2sE^n}{db^{n-2}}$, we have $\dot{E} = \frac{2sE^n}{db^{n-2}}$, or $\frac{2sE^n\dot{x}}{db^{n-2}}$, and therefore $\dot{x} = \frac{db^{n-2}\dot{E}}{2sE^n}$; whence $x = \frac{db^{n-2}}{2s} \times \frac{E^{1-n} - a^{1-n}}{n-1}$. *Q. E. I.*

## C O R O L. I.

THE Difference of the two Arcs deſcribed, in the Deſcent and ſubſequent Aſcent, is in the Ratio of the ſame Power of either of thoſe Arcs, as the Reſiſtance is of the Velocity.

## COROL. II.

THE Refiftance being in the duplicate Ratio of the Celerity, and the Lengths of the two Arcs given; the Number of Ofcillations betwixt the Times of defcribing thofe Arcs will continue the fame very nearly, let the Cycloid, or Length of the Pendulum, be what it will.

## COROL. III.

IF the Refiftance be either uniform, or directly in the Ratio of any Power of the Velocity lefs than in the fimple Ratio, as the fubduplicate, fubtriplicate, &c. the Body will continue vibrating 'till it hath compleated $\frac{d a^{1-n}}{2 s b^{2-n} \times \overline{1-n}}$ entire Ofcillations, and then will have entirely loft all its Motion.

## COROL. IV.

WHEN the Refiftance is in more than the fimple Ratio of the Velocity, the Motion will be prolonged *ad infinitum.*

## COROL. V.

LASTLY, if any two Arcs of the Cycloid, or fmall Arcs of a Circle, be taken in a given Ratio to each other, the Number of Vibrations performed between the Times of defcribing thofe Arcs, in one whole Defcent of the Pendulum, will be nearly in the inverfe Ratio of that Power of either of the faid Arcs, whofe Exponent is lefs by Unity than that expreffing the Ratio of the Refiftance; that

that is, if two Arcs, A, B, be taken in the same Ratio, as two other Arcs, C, D, the Number of Vibrations betwixt describing the two former, will be to the Number betwixt describing the two latter, in one whole Descent of the Pendulum, as $C^{n-1}$ to $A^{n-1}$, or as $D^{n-1}$ to $B^{n-1}$. From whence, and the foregoing Conclusions, not only the Law, but the absolute Resistance of Mediums may be found, by observing the Number of Vibrations performed therein by given Pendulums, in losing given Parts of their Motion.

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## A new Method for the Solution of Equations in Numbers.

### CASE I.

*When only one Equation is given, and one Quantity (x) to be determined.*

TAKE the Fluxion of the given Equation (be it what it will) supposing, $x$, the unknown, to be the variable Quantity; and having divided the whole by $\dot{x}$, let the Quotient be represented by A. Estimate the Value of $x$ pretty near the Truth, substituting the same in the Equation, as also in the Value of A, and let the Error, or resulting Number in the former, be divided by this numerical Value of A, and the Quotient be subtracted from the said former Value of $x$; and from thence will arise a new Value of that Quantity much nearer to the Truth than the former, wherewith proceeding as before, another new Value may be had, and so another, &c. 'till we arrive to any Degree of Accuracy desired.

## CASE II.

*When there are two Equations given, and as many Quantities ($x$ and $y$) to be determined.*

TAKE the Fluxions of both the Equations, confidering $x$ and $y$ as variable, and in the former collect all the Terms, affected with $\dot{x}$, under their proper Signs, and having divided by $\dot{x}$, put the Quotient $= A$; and let the remaining Terms, divided by $\dot{y}$, be reprefented by B: In like manner, having divided the Terms in the latter, affected with $\dot{x}$, by $\dot{x}$, let the Quotient be put $= a$, and the reft, divided by $\dot{y}$, $= b$. Affume the Values of $x$ and $y$ pretty near the Truth, and fubftitute in both the Equations, marking the Error in each, and let thefe Errors, whether pofitive or negative, be fignified by R and $r$ refpectively: Subftitute likewife in the Values of A, B, $a$, $b$, and let $\frac{Br - bR}{Ab - aB}$ and $\frac{aR - Ar}{Ab - aB}$ be converted into Numbers, and refpectively added to the former Values of $x$ and $y$; and thereby new Values of thofe Quantities will be obtained; from whence, by repeating the Operation, the true Values may be approximated *ad libitum*.

*Note*, 1. That every Equation is firft to be fo reduced by Tranfpofition, that the Whole may be equal to Nothing.

2. That, if after the firft Operation, the Value of $x$ or $y$ be not found to come out pretty nearly as affumed, fuch Value is not to be depended on, but a new Eftimation made, and the Operation begun again.

3.

3. That, the above Method, for the general part, when $x$ and $y$ are near the Truth, doubles the Number of Places at each Operation, and only converges ſlowly, when the Diviſor A, $Ab - aB$, at the ſame time converges to nothing.

## EXAMPLE I.

LET $300x - x^3 - 1000$ be given $= 0$; to find a Value of $x$. From $300\dot{x} - 3x^2\dot{x}$, the Fluxion of the given Equation, having expunged $\dot{x}$, (*Caſe* I.) there will be $300 - 3xx = A$: And, becauſe it appears by Inſpection, that the Quantity $300x - x^3$, when $x$ is $= 3$, will be leſs, and when $x = 4$, greater than 1000, I eſtimate $x$ at 3.5; and ſubſtitute inſtead thereof, both in the Equation and in the Value of A, finding the Error in the former $= 7.125$, and the Value of the latter $= 263.25$: Wherefore, by taking $\frac{7.125}{263.25} = .027$ from 3.5 there will remain 3.473 for a new Value of $x$; with which proceeding as before, the next Error, and the next Value of A, will come out .00962518, and 263.815 reſpectively; and from thence the third Value of $x = 3.47296351$; which is true, at leaſt, to 7 or 8 Places.

## EXAMPLE II.

LET $\sqrt{1-x} + \sqrt{1-2xx} + \sqrt{1-3x^3} - 2 = 0$. This in Fluxions will be $\frac{-\dot{x}}{2\sqrt{1-x}} - \frac{2x\dot{x}}{\sqrt{1-2xx}} - \frac{9x^2\dot{x}}{2\sqrt{1-3x^3}}$, and therefore A, here, $= -\frac{1}{2\sqrt{1-x}} - \frac{2x}{\sqrt{1-2xx}} - \frac{9x^2}{2\sqrt{1-3x^3}}$; wherefore if $x$ be ſuppoſed $= .5$, it will become $-3.545$: And, by ſubſtituting 0.5 inſtead of $x$ in the given

given Equation, the Error will be found .204; therefore $\frac{.204}{-3.545}$ (equal $-$ .057) ſubtracted from .5, gives .557 for the next Value of $x$; from whence, by proceeding as before, the next following will be found .5516, &c.

## EXAMPLE III.

LET there be given the Equations $y + \sqrt{y^2 - x^2} - 10 = 0$, and $x + \sqrt{yy + x} - 12 = 0$; to find $x$ and $y$.

The Fluxions here being $\dot{y} + \frac{y\dot{y} - x\dot{x}}{\sqrt{yy - xx}}$ and $\dot{x} + \frac{y\dot{y} + \frac{1}{2}\dot{x}}{\sqrt{yy + x}}$ or $\dot{y} + \frac{y\dot{y}}{\sqrt{yy - xx}} - \frac{x\dot{x}}{\sqrt{yy - xx}}$, and $\dot{x} + \frac{\frac{1}{2}\dot{x}}{\sqrt{yy + x}} + \frac{y\dot{y}}{\sqrt{yy + x}}$, we have A equal $-\frac{x}{\sqrt{yy - xx}}$, B equal $1 + \frac{y}{\sqrt{yy + xx}}$, $a = 1 + \frac{\frac{1}{2}}{\sqrt{yy + x}}$, and $b = \frac{y}{\sqrt{yy + x}}$ (*Caſe* II.)

Let $x$ be ſuppoſed equal 5, and $y$ equal 6; then will R equal $-$ .68, $r$ equal $-$ .6, A equal $-$ 1.5, B equal 2.8, $a$ equal 1.1, $b$ equal 9; therefore $\frac{\mathrm{B}r - b\mathrm{R}}{\mathrm{A}b - a\mathrm{B}} = .23$, and $\frac{a\mathrm{R} - \mathrm{A}r}{\mathrm{A}b - a\mathrm{B}} = .37$, and the new Values of $x$ and $y$ equal to 5.23, and 6.37 reſpectively; which are as near the Truth as can be exhibited in three Places only, the next Values coming out 5.23263 and 6.36898.

*Note*, When Equations are given to be ſolved in this manner, it will be convenient, that they be firſt of all reduced to the moſt commodious Forms, to facilitate the Operations, whether into Fractions or Surds, or *vice verſa*: For Inſtance, the Equations in the laſt Example had been much

much eaſier ſolved, had they been firſt reduced, out of Surds, to $20y - xx - 100 = 0$, and $yy - xx + 25x - 144$ equal 0, or, by exterminating $y$, and working according to *Caſe* I. whereas, on the other hand, to have reduced the Equation, in the preceding Example, out of Surds (as is uſual in other Methods) would have rendered the Trouble of Solution almoſt inſuperable.

## EXAMPLE. IV.

LET $49 \times \overline{x - \frac{x}{x+y}}|^{\frac{1}{2}} - 25 \times \overline{1 - \frac{xx}{1+y}}|^{\frac{1}{2}} = 0$, and $81 \times \overline{1 - \frac{xx}{1+y}}|^{\frac{1}{2}} - 49 \times \overline{1\ \frac{x}{y} - \frac{xy}{1+x}}|^{\frac{1}{2}} = 0$.

Here, taking the Fluxions of both the Equations, and proceeding according to *Caſe* II. we have A equal $49 \times \overline{1 + \frac{x-y}{x+y}}|^{\frac{1}{2}} + \frac{50x}{\overline{1+y}|^{\frac{1}{2}}}$, $\mathrm{B} = \frac{98x}{\overline{x+y}|^{\frac{1}{2}}} - \frac{50x^2}{\overline{1+y}|^{\frac{3}{2}}}$, $a = \frac{-162x}{\overline{1+y}|^{\frac{1}{2}}} + 49 \times \overline{\frac{y}{1+x} - \frac{1}{y} - \frac{2xy}{\overline{1+x}|^{2}}}$, and $b = \frac{162x^2}{\overline{1+y}|^{\frac{3}{2}}} + 49x \times \overline{\frac{1}{yy} + \frac{1}{\overline{1+x}|^{2}}}$

Suppoſe $x = .8$, and $y = .6$; then will be found $\mathrm{R} = .45$, $r = 2.66$, $\mathrm{A} = 68$, $\mathrm{B} = 20.7$, $a = -131$, $b = 146$, and the next Values of $x$ and $y$ equal to .799 and .582; with which, repeating the Operation, the next following will come out .79912 and .58138, both which are true, at leaſt, to 4 Places: But, if a greater Exactneſs ſhould be deſired, let the Operation be once more repeated, and then the next Values will be true to double thoſe Places.

*N. B.* Altho' in ſeveral Caſes it happens, that the required Values, from the Equations themſelves, cannot be aſſumed

 near

near the Truth without ſome Attention and Trouble; yet, from the Nature of the *Problem* from whence thoſe Equations are derived, when that is known, the Trouble may be avoided, and the Thing effected without any great Difficulty: For inſtance, tho' it is not eaſy to perceive, that $y$ and $x$ are about $\frac{6}{10}$ and $\frac{8}{10}$ in the laſt Example; yet, when it is known, that 1, $x$, and $y$, are the Sides of a Plain Triangle, wherein Lines, drawn to biſect each Angle and terminate in thoſe Sides, are to one another, reſpectively, as 5, 7, and 9, the Thing then appears evident upon the firſt Conſideration.

## EXAMPLE V

LET $x^x + y^y - 1000 = 0$, and $x^y + y^x - 100 = 0$. Here we ſhall have $A = \overline{1 + L : x} \times x^x$, B equal $\overline{1 + L : y} \times y^y$, $a = \frac{y}{x} \times x^y + y^x L : y$, and $b$ equal $\frac{x}{y} \times y^x + x^y L : x$. Now, it appearing from the firſt Equation, that the greateſt of the two required Quantities cannot be leſſer than 4, nor greater than 5; and from the firſt and ſecond together, that the Difference of $x$ and $y$ muſt be pretty large; otherwiſe $x^x + y^y$ could not be 10 times as great as $x^y + y^x$: I therefore take $x$ (which I ſuppoſe the greater Number) equal 4.5, and $y$ equal 2.5; and then by a Table of Logarithms, or otherwiſe, find the next Values of theſe Quantities to be 4.55 and 2.45; and the next following 4.5519, *&c.* and 2.4495, *&c.* reſpectively.

# Of INCREMENTS.

## PROPOSITION I.

*If,* $\underset{''}{n}$, $\underset{'}{n}$, $n$, $\overset{'}{n}$, $\overset{''}{n}$, $\overset{'''}{n}$, &c. *be a Series of Terms in a decreasing Arithmetical Progression, whose common Difference is* $\dot{n}$; *and* $\overset{'}{n}\,\overset{''}{n}\,\overset{'''}{n}\ldots\ldots\times\overset{r}{\overbrace{n}}$, *a Product arising from the Multiplication of any Number,* r, *of those Terms, immediately succeeding each other, continually together; and if each of the Factors in this Product be increased by the common Difference: I say, the Product it self will be increased by* $r\dot{n}\times\overset{'}{n}\,\overset{''}{n}\,\overset{'''}{n}\ldots\ldots\times\overset{r-1}{\overbrace{n}}$; *taken under the first Values of those Quantities.*

FOR, since $\overset{'}{n}$, increased by the common Difference, becomes equal to $n$; and $\overset{''}{n}$, increased by the same Difference, equal to $\overset{'}{n}$, *&c. &c.* it is manifest, that the new Value of the said Product, arising from such an Increase of its Factors, will be equal to $n\,\overset{'}{n}\,\overset{''}{n}\ldots\ldots\times\overset{r-1}{\overbrace{n}}$, under the first Values of those Quantities, from which taking the former, or given Value of that Product, we have $n\,\overset{'}{n}\,\overset{''}{n}\ldots\times\overset{r-1}{\overbrace{n}} - \overset{'}{n}\,\overset{''}{n}\,\overset{'''}{n}\ldots\times\overset{r}{\overbrace{n}} = \overset{'}{n}\,\overset{''}{n}\,\overset{'''}{n}\ldots\times\overset{r-1}{\overbrace{n}}$ into $n - \overset{r}{\overbrace{n}}$ for the Increment; which, because the Excess of $n$ above $\overset{r}{\overbrace{n}}$ is equal to $r$ times the common Difference, will consequently be equal to $r\dot{n}\times\overset{'}{n}\,\overset{''}{n}\,\overset{'''}{n}\ldots\ldots\overset{r-1}{\overbrace{n}}$. *Q. E. D.*

COROL-

## COROLLARY.

SINCE the Increment of $\overset{\prime}{n}\,\overset{\prime\prime}{n}\,\overset{\prime\prime\prime}{n} \ldots\ldots \times \overset{\underset{\smile}{r}}{n}$ is proved to be $= r\dot{n} \times \overset{\prime}{n}\,\overset{\prime\prime}{n}\,\overset{\prime\prime\prime}{n} \ldots\ldots \times \overset{\underset{\smile}{r-1}}{n}$, that of $\dfrac{\overset{\prime}{n}\,\overset{\prime\prime}{n}\,\overset{\prime\prime\prime}{n} \ldots\ldots \times \overset{\underset{\smile}{r}}{n}}{r\dot{n}}$ must consequently be $\overset{\prime}{n}\,\overset{\prime\prime}{n}\,\overset{\prime\prime\prime}{n} \ldots\ldots \times \overset{\underset{\smile}{r-1}}{n}$: Whence, to find a Product of this kind from its Increment given, the following Rule is derived, *i. e.* Increase the Number of Factors, by annexing to them the next inferior Term of the Progression, and divide the whole by the Number of Factors, thus increased, drawn into the common Difference.

*Note*, That $\overset{\underset{\smile}{r-1}}{n}$ stands for the Term of the proposed Progression, whose Distance from $n$ is $r-1$, when on the descending Side, and $\overset{\underset{\frown}{r-1}}{n}$ when on the ascending Side; the like is to be understood of any other.

## EXAMPLE I.

LET a Product or Quantity, expressing the Value of $1+2+3+4+5+\ldots\ldots n$ be required; or, which is the same in effect, let it be required to find an Expression so affected, that increasing $n$ by $1$ (or writing therein $n+1$ instead of $n$) it shall be augmented by $n+1$. Then, if $\underset{,}{n}$, $n$, $\overset{\prime}{n}$, *&c.* be assumed for a Series of Numbers in Arithmetical Progression, whose common Difference is $1$, according to the above Notation, we shall have $\underset{,}{n}$ equal to the given Increment in this Case; to which annexing $n$, the next inferior Term of the Progression, and dividing the Product $(\underset{,}{n}\,n)$ by $(2)$ the Number of Factors, drawn into the com-

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mon Difference; there comes out $\frac{1}{2}\underset{\prime}{n}n$, or $\frac{\overline{n+1}\times n}{2}$, which is equal $1+2+3+4+5\ldots+n$, the Value propofed.

EXAMPLE II.

LET it be required to find the Sum of a Series of Cubes, as $1+8+27+64+125$, &c. Put $n$ for the Number of Cubes to be taken, S their required Sum, and $\dot{S}$ the next fucceeding Cube of the Series after the laft in S, or the Increment of S that will arife by augmenting $(n)$ the Number of Terms by Unity: Then, becaufe 1 is equal to the Root of the firft Term, and alfo equal to the common Difference, the Root of the Term $\dot{S}$, it is manifeft, will be $= n+1$; or, if $\underset{\prime\prime}{n}, \underset{\prime}{n}, n, \overset{\prime}{n}$, &c. be put for a Series of Numbers whofe common Difference is 1, equal to $n$; wherefore $\dot{S}$ is equal $\underset{\prime}{n}\underset{\prime}{n}\underset{\prime}{n}$. But to bring this Value of $\dot{S}'$ to the Form of the Propofition; inftead thereof, let its Equal $\overline{\underset{\prime\prime}{n}-1}\times\underset{\prime}{n}\times\overline{n+1}$, or $\underset{\prime\prime}{n}\underset{\prime}{n}n+\underset{\prime}{n}$ be fubftituted; then S, according to the Rule, will be $\frac{\underset{\prime\prime}{n}\underset{\prime}{n}n\overset{\prime}{n}}{4}+\frac{\underset{\prime}{n}n}{2}$ where, for $\underset{\prime\prime}{n}\overset{\prime}{n}$, writing its Equal $n\underset{\prime}{n}-2$, it will become $S=\frac{\overline{\underset{\prime}{n}n}|^2}{4}=\frac{\overline{n+1\times n}|^2}{4}$

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## EXAMPLE III.

To find the Unciæ of a Binomial raifed to a given Power.

Let $a+b$ be the propofed Binomial, $n$ the Exponent of its Power, and let B, C, &c. be the required Unciæ, or, which is the fame in effect, let $\overline{a+b}|^n$ be $= a^n + B\,a^{n-1} \times b + C a^{n-2}\, b^2 + D a^{n-3}\, b^3$, &c.

The Equation multiplied by $a+b$ becomes $\overline{a+b}|^{n+1} = a^{n+1} + \left.{B \atop 1}\right\}\, a^n b + \left.{C \atop B}\right\}\, a^{n-1} b^2 + \left.{D \atop C}\right\}\, a^{n-2} b^3 + \left.{E \atop D}\right\}\, a^{n-3} b^4$ &c. wherefore, becaufe the Unciæ of the Power, whofe Exponent is $\left\{ {n \atop n+1} \right\}$ are $\left\{ {1,\ B,\ C,\ D,\ E,\ F,\ G,\ \&c. \atop 1,\ B+1,\ C+B,\ D+C,\ E+D,\ F+E,\ G+F,\ \&c.} \right.$ it is manifeft, that the Values of B, C, D, &c. are fuch, that increafing ($n$) the Exponent by Unity, they will be increafed by 1, B, C, &c. refpectively. But the Increment of B being $= 1$, the Value of B, or Increment of C, will be $= n$; and therefore C, or the Increment of D, equal to $\frac{n\,\acute{n}}{2}$; therefore D equal to $\frac{n\,\acute{n}\,\acute{\acute{n}}}{2.3}$; therefore F equal to $\frac{n\,\acute{n}\,\acute{\acute{n}}\,\acute{\acute{\acute{n}}}}{2.3.4}$ &c. &c. where $\acute{n}$, $\acute{\acute{n}}$, $\acute{\acute{\acute{n}}}$, &c. ftand for $n-1$, $n-2$, $n-3$, &c. refpectively.

PROP.

## PROPOSITION II.

*Suppoſing* $\dot{n}$, $\underset{\prime\prime}{n}$, $\underset{\prime}{n}$, $n$, $\overset{\prime}{n}$, &c. *to be as in the laſt* Propoſition: *I ſay, if each Factor in the Denominator of the Fraction* $\frac{1}{n \times \underset{\prime}{n} \times \underset{\prime\prime}{n} \times \underset{\prime\prime\prime}{n} \ldots\ldots \underset{\overset{\frown}{r-1}}{n}}$ *be diminiſhed by the common Difference, the Fraction it ſelf will be increaſed by* $\frac{r\dot{n}}{\overset{\prime}{n}\, n\, \underset{\prime}{n}\, \underset{\prime\prime}{n} \ldots\ldots \underset{\overset{\frown}{r-1}}{n}}$.

FOR, ſince $n$, diminiſhed by the common Difference, becomes equal to $\overset{\prime}{n}$; and $\underset{\prime}{n}$, diminiſhed by the ſame Difference, equal to $n$, *&c.* the new Value of the ſaid Fraction, ariſing from ſuch a Diminution of its Factors, will, it is evident, be equivalent to $\frac{1}{\overset{\prime}{n}\, n\, \underset{\prime}{n}\, \underset{\prime\prime}{n} \ldots\ldots \underset{\overset{\frown}{r-2}}{n}}$, under the firſt or given Values of thoſe Quantities; and therefore the Increaſe thereof muſt be $\frac{1}{\overset{\prime}{n}\, n\, \underset{\prime}{n} \ldots\ldots \underset{\overset{\frown}{r-2}}{n}} - \frac{1}{n\, \underset{\prime}{n}\, \underset{\prime\prime}{n} \ldots\ldots \underset{\overset{\frown}{r-1}}{n}}$ equal to

$$\frac{1}{n\, \underset{\prime}{n}\, \underset{\prime\prime}{n} \ldots\ldots \underset{\overset{\frown}{r-2}}{n}} \times \overline{\frac{1}{\overset{\prime}{n}} - \frac{1}{\underset{\overset{\frown}{r-1}}{n}}} = \frac{1}{n\, \underset{\prime}{n}\, \underset{\prime\prime}{n} \ldots\ldots \underset{\overset{\frown}{r-2}}{n}} \times \frac{\underset{\overset{\frown}{r-1}}{n} - \overset{\prime}{n}}{\overset{\prime}{n} \times \underset{\overset{\frown}{r-1}}{n}}$$ equal to

$\underset{\overset{\frown}{r-1}}{n}$

$$\frac{n_{\overset{\frown}{r-1}} - \overset{\prime}{n}}{\overset{\prime}{n}\, n\, n_{\prime}\, n_{\prime\prime} \ldots\ldots n_{\overset{\frown}{r-1}}} = \frac{r\dot{n}}{\overset{\prime}{n}\, n\, n_{\prime}\, n_{\prime\prime} \ldots\ldots n_{\overset{\frown}{r-1}}}$$, becaufe $n_{\overset{\frown}{r-1}} - \overset{\prime}{n}$ equal to $r\dot{n}$.

*Q. E. D.*

## COROLLARY.

WHEREFORE, the Quantity, or Fraction, whofe Increment is $\frac{r\dot{n}}{\overset{\prime}{n}\, n\, n_{\prime}\, n_{\prime\prime} \ldots\ldots n_{\overset{\frown}{r-1}}}$ being $\frac{1}{n\, n_{\prime}\, n_{\prime\prime}\, n_{\prime\prime\prime} \ldots\ldots n_{\overset{\frown}{r-1}}}$, the Quantity whofe Increment is $\frac{1}{\overset{\prime}{n}\, n\, n_{\prime} \ldots\ldots n_{\overset{\frown}{r-1}}}$ muft confequently be $\frac{1}{r\dot{n} \times n\, n_{\prime}\, n_{\prime\prime} \ldots n_{\overset{\frown}{r-1}}}$: Whence to find the Value of a Fraction of this kind, from its Increment given, there arifes this *Rule*. Strike out the leaft Factor in the Denominator of the given Increment, and inftead thereof put the Rectangle of the common Difference of the Factors into the Number of remaining Factors; the Refult will be the Value that was to be found.

## EXAMPLE.

LET it be required to find the Sum of the infinite Series $\frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6}$, *&c.* or one fingle Frac-

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tion (if poffible) that fhall exprefs the Value of, &c. $\frac{1}{4 \cdot 5} + \frac{1}{3 \cdot 4} + \frac{1}{2 \cdot 3}$. Here, if the required Fraction be confidered as made up, or generated by a continual and regular Addition of the Terms, &c. $\frac{1}{4 \cdot 5}$, $\frac{1}{3 \cdot 4}$, $\frac{1}{2 \cdot 3}$ of the propofed Series, then $\frac{1}{1 \cdot 2}$ being the next fucceeding Term of the Progreffion, or the Increment of the faid Fraction, the Fraction it felf, by the foregoing Rule, will be $= \frac{1}{1 \times \overline{2 - 1} \times 2} = \frac{1}{2}$

In like manner will be found

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5}, \text{ &c.} = 1.$$

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5}, \text{ &c.} = \frac{1}{2 \cdot 2}.$$

$$\frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{3 \cdot 4 \cdot 5 \cdot 6}, \text{ &c.} = \frac{1}{2 \cdot 3 \cdot 3}.$$

$$\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{1}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}. \text{ &c.} = \frac{1}{2 \cdot 3 \cdot 4 \cdot 4}.$$

$$\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{1}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}, \text{ &c.} = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 5}.$$

*N. B.* That in finding the Value of any Quantity by the Methods foregoing, it ought to be well confidered, from the Nature of the Queftion, whether that Quantity confifts barely of fuch Fraction, Product, or Products, as are fpecified in the Propofitions, or thofe joined to fome invariable Quantity (as is done in Fluxions) and Allowances is to be made accordingly.

*An Investigation of Sir* Isaac Newton's *Theorem for finding the Sum of a Series of Numbers by means of their Differences.*

LET $a, b, c, d, e, f, g, h,$ *&c.* be the given Series of Numbers: Then, by taking each of them from the next succeeding, there will be $-a+b$, $-b+c$, $-c+d$, $-d+e$, $-f+g$, for the first Differences: Again, taking each of these Differences from its succeeding one, we have $a-2b+c$, $b-2c+d$, $c-2d+e$, $d-2e+f$, *&c.* for the second Differences. In like manner the third Differences will be found $-a+3b-3c+d$, $-b+3c-3d+e$, $-c+3d-3e+f$, *&c.* and the fourth, $a-4b+6c-4d+e$, $b-4c+6d-4e+f$, *&c. &c.* Let the first Difference of the first Order be called $\overset{\prime}{D}$, the first of the second Order $\overset{\prime\prime}{D}$, the first of the third Order $\overset{\prime\prime\prime}{D}$, *&c.* then we shall have $a=a$, $b=a+\overset{\prime}{D}$, $c=-a+2b+\overset{\prime\prime}{D}$, $d=a-3b+3c+\overset{\prime\prime\prime}{D}$, *&c.* and from thence by Substitution,

$$a=a$$
$$b=a+\overset{\prime}{D}$$
$$c=a+2\overset{\prime}{D}+\overset{\prime\prime}{D}$$
$$d=a+3\overset{\prime}{D}+3\overset{\prime\prime}{D}+\overset{\prime\prime\prime}{D}$$
$$e=a+4\overset{\prime}{D}+6\overset{\prime\prime}{D}+4\overset{\prime\prime\prime}{D}+\overset{\prime\prime\prime\prime}{D}.$$

where the Law of Continuation is manifest, the Unciæ of the Values of $c$, $d$, $e$, *&c.* being those of a Binomial, raised to the second, third, fourth Powers, *&c.* Therefore, if $n$ be put for the Number of Terms in the proposed Series $a+b+c+d$, *&c.* whose Sum we are about to find, the Value of

of the next Term in the Progression after the last in that Series, or that whose Place is defined by $n+1$, will, it is plain, be equal to $a + n\dot{D} + n \times \frac{n-1}{2} D'' + n \times \frac{n-1}{2} \times \frac{n-2}{3} D''' + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} D''''$, &c. And so much will the said Sum be increased by augmenting $n$, the given Number of Terms by Unity; which Sum, therefore, by the first of the two foregoing *Propositions*, is $na + n \times \frac{n-1}{2} D' + n \times \frac{n-1}{2} \times \frac{n-2}{3} D'' + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} D'''$, &c. *Q. E. I.*

## EXAMPLE.

SUPPOSE $a$, $b$, $c$, $d$, &c. to be a Series of Squares, as 9, 16, 25, 36, &c. whose Roots are in Arithmetical Progression; then will the first Differences be 7, 9, 11, 13, &c. the second, 2, 2, 2, &c. the third, 0, 0, &c. &c. Therefore $a = 9$, $D = 7$, $D' = 2$, $D'' = 0$, $D''' = 0$, &c. and consequently $na + n \times \frac{n-1}{2} D'$, &c. equal $9n + n \times \frac{n-1}{2} \times 7 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times 2 = 9 + 16 + 25 + 36 + 49$, &c. continued to $n$ Terms.

In the same manner the Sum of a Series of *Cubes*, *Biquadrates*, &c. may be found.

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## An eafy and general Manner of inveftigating the Sum of any recurring Series.

### PROPOSITION.

*Suppofing $p$, $q$, $r$, $s$, &c. to be any Quantities, either pofitive or negative, and* $A+B+C+D+E$, &c. *a recurring Series, or one whofe Terms* A, B, C, &c. *are fo related, that any one of them, being multiplied by $p$, the next following, by $q$, the next in Order, by $r$,* &c. *the Sum of all the Products, thus arifing, fhall be equal to* o: *To find* $(x)$ *the Sum of fuch a Series.*

BECAUSE, by Suppofition, $pA+qB+rC$, &c. is equal o, $pB+qC+rD$, &c. equal o, $pC+qD+rE$, &c. equal o, &c. &c. it is evident, that the Sum of

$$\text{all thefe, or,}\left\{\begin{array}{l} pA+qB+rC+sD, \&c. \\ pB+qC+rD+sE, \&c. \\ pC+qD+rE+sF, \&c. \\ pD+qE+rF+sG, \&c. \\ pE+qF+rG+sH, \&c. \\ \&c. \quad \&c. \quad \&c. \quad \&c. \quad \&c. \end{array}\right\}\text{muft confequently be equal o.}$$

Where, becaufe $A+B+C+D$, &c. is $=x$, $B+C+D$, &c. $=x-A$, $C+D+E$, &c. $=x-A-B$, &c. &c. the Value of the firft Column towards the Left-hand will be $px$; that of the fecond, $q\times\overline{x-A}$; that of the third, $r\times\overline{x-A-B}$; that of the fourth, $s\times\overline{x-A-B-C}$, &c. &c. and therefore

$px$

$px + q \times \overline{x - \mathrm{A}} + r \times \overline{x - \mathrm{A} - \mathrm{B}} + s \times \overline{x - \mathrm{A} - \mathrm{B} - \mathrm{C}}$, &c. or $px + qx - q\mathrm{A} + rx - r \times \overline{\mathrm{A} + \mathrm{B}} + sx - s \times \overline{\mathrm{A} + \mathrm{B} + \mathrm{C}}$, &c. $= 0$; hence $x = \frac{q\mathrm{A} + r \times \overline{\mathrm{A} + \mathrm{B}} + s \times \overline{\mathrm{A} + \mathrm{B} + \mathrm{C}} + t \times \overline{\mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D}}}{p + q + r + s + t + v, \text{ \&c.}}$, &c. *Q. E. I.*

## EXAMPLE I.

LET the Series $ay + y^2 + \frac{y^3}{a}$, &c. $(= x)$ be propofed, where $p$ equal $-y$, $q = a$, $r = 0$, $s = 0$, &c. $\mathrm{A} = ay$, $\mathrm{B} = yy$, &c, then will $x = \frac{q\mathrm{A} + r \times \overline{\mathrm{A} + \mathrm{B}}, \text{ \&c.}}{p + q, \text{ \&c.}}$, become $\frac{a^2 y}{a - y}$ in this Cafe.

## EXAMPLE II.

LET $1 - \frac{3}{z^3} + \frac{5}{z^6} - \frac{7}{z^9} + \frac{9}{z^{12}}$, &c. $= x$, where $p = 1$, $q = 2z^3$, $r = z^6$, $s = 0$, $t = 0$, &c. Then $\mathrm{A}$ being $= 1$, $\mathrm{B} = -\frac{3}{z^3}$, &c. $\frac{q\mathrm{A} + r \times \overline{\mathrm{A} + \mathrm{B}}, \text{ \&c.}}{p + q, \text{ \&c.}}$ $(= x)$ will here become $\frac{z^6 - z^3}{z^6 + 2z^3 + 1}$.

## A Method for finding the Sum of a Series of Powers, &c.

### PROPOSITION

*To find the Sum of any Series of Powers whose Roots are in Arithmetical Progression, as* $\overline{m+d}|^n + \overline{m+2d}|^n + \overline{m+3d}|^n \ldots\ldots\ldots x^n$, *m, d, and n, being any Numbers whatsoever.*

LET $Ax^{n+1} + Bx^n + Cx^{n-1} + Dx^{n-2} + Ex^{n-3} + Fx^{n-4}$, &c. $-$ K, if possible, be always equal to $\overline{m+d}|^n + \overline{m+2d}|^n \ldots\ldots x^n$, and A, B, C, &c. determinate Quantities: Then, if any other Number in the Progression $m+d$, $m+2d$, $m+3d \ldots\ldots x+d$, $x+2d$, $x+3d$, &c. as $x+d$, be substituted instead of $x$, the Equality will still continue; and we shall have $A\times\overline{x+d}|^{n+1} + B\times\overline{x+d}|^n + C\times\overline{x+d}|^{n-1} + D\times\overline{x+d}|^{n-2}$ &c. $-$ K equal $\overline{m+d}|^n + \overline{m+2d}|^n \ldots\ldots\ldots \overline{x+d}|^n$; from which taking the former Equation there remains $A\times\overline{x+d}|^{n+1} - x^{n+1} + B\times\overline{x+d}|^n - x^n + C\times\overline{x+d}|^{n-1} - x^{n-1}$, &c. $= \overline{x+d}|^n$, shewing how much each Side is increased by augmenting the Number of Terms in the given Series by Unity; where, by transposing $\overline{x+d}|^n$, and throwing the several Powers of $x+d$ into Series, we have

$-x$

$$\left.\begin{array}{llll}
-x^{n} & -n x^{n-1} d & -n\times\frac{n-1}{2}\times x^{n-2} d^{2} & -n\times\frac{n-1}{2}\times\frac{n-2}{3} x^{n-3} d^{3}, \&c. \\
\overline{n+1} A x^{n} d + & \overline{n+1}\times\frac{n}{2} A x^{n-1} d^{2} + & \overline{n+1}\times\frac{n}{2}\times\frac{n-1}{3} A x^{n-2} d^{3} + & \overline{n+1}\times\frac{n}{2}\times\frac{n-1}{3}\times\frac{n-2}{4} A x^{n-3} d^{4}, \&c. \\
* & + n B x^{n-1} d^{2} + & n\times\frac{n-1}{2} B x^{n-2} d^{3} + & n\times\frac{n-1}{2}\times\frac{n-2}{3} B x^{n-3} d^{4}, \&c. \\
* & * & + \overline{n-1}\times C x^{n-2} d^{3} + & \overline{n-1}\times\frac{n-2}{2} C x^{n-3} d^{4}, \&c. \\
* & * & * & \overline{n-2} D x^{n-3} d^{4}, \&c. \\
 & & & \&c.
\end{array}\right\} = 0.$$

From which, by equating the homologous Terms, A will come out $= \frac{1}{\overline{n+1} \times d}$, $B = \frac{1}{2}$, $C = \frac{nd}{3.4}$, $D = 0$, $E = -\frac{n \times \overline{n-1} \times \overline{n-2} \times d^3}{2.3.4.5.6}$, $F = 0$, $G = \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times d^5}{2.3.4.5.6.7.6}$ $H = 0$, &c. wherefore the Values of A, B, C, &c. being so assigned, the whole Expression, or its Equal $-\overline{x+d}|^n$ $+ A \times \overline{\overline{x+d}|^{n+1} - x^{n+1}} + B \times \overline{\overline{x+d}|^n - x^n}$, &c. must be equal o, and consequently $A \times \overline{\overline{x+d}|^{n+1} - x^{n+1}} + \overline{B \times \overline{x+d}|^n - x^n}$, &c. $= \overline{x+d}$; that is, let $x$ and $n$ be what they will, the foresaid Increments of $A x^{n-1} + B x^n + C x^{n-1}$, &c. $-$ K and $\overline{m+d}|^n + \overline{m+2d}|^n$, &c. will, under the above assigned Values of A, B, &c. be equal to one another: Therefore, if K be taken equal $A m^{n+1} + B m^n + C m^{n-1}$, &c. so that when $x$ equal $m$, or the proposed Series is equal to Nothing, $A x^{n+1} + B x^n$, &c. $-$ K may be also $= 0$, it is manifest, that these two Expressions, as they are increased alike, will, in all other Circumstances, be equal; that is, let $x$ be what it will $A x^{n+1} + B x^n + C x^{n-1} + D x^{n-2}$, &c. $- A m^{n+1} - B m^n - C m^{n-1} - D m^{n-2}$, &c. under the said Values of A, B, C, &c. will be always equal to $\overline{m+d}|^n + \overline{m+2d}|^n + \overline{m+3d}|^n \ldots x^n$; which Values being therefore substituted, there will be $\frac{x^{n+1}}{\overline{n+1} \times d} + \frac{x^n}{2} + \frac{d n x^{n-1}}{3.4} - \frac{n \times \overline{n-1} \times \overline{n-2} d^3 x^{n-3}}{2.3.4.5.6}$

+

$$+\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}}{2.3.4.5.6.7.6}d^5x^{n-5}-\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3},\ \&c.}{2.3.4.5.6.7.8.5.6.}d^7x^{n-7}$$

$$+\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\overline{n-5}\times\overline{n-6}\times\overline{n-7}\times\overline{n-8}}{2.3.4.5.6.7.8.9.11.12.}d^9x^{n-9},\ \&c.\ -$$

$$\frac{m^{n+1}}{\overline{n+1}\times d}-\frac{m^n}{2}-\frac{ndm^{n-1}}{3.4}+\frac{n\times\overline{n-1}\times\overline{n-2}}{2.3.4.5.6}\times d^3m^{n-3},\ \&c.$$

$$=\overline{m+d}|^n+\overline{m+2d}|^n+\overline{m+3d}|^n\ \ldots\ldots\ x^n$$

*Q. E. I.*

## COROL. I.

HENCE, if $n$ be a whole poſitive Number, and $m$ be taken equal o; then all the Terms in the ſecond Series $-\frac{m^{n+1}}{\overline{n+1}\times d}-\frac{m^n}{2}+\frac{ndm^{n-1}}{3.4}$, &c. vaniſhing when $n$ is even, and all but that where the Exponent of $m$ is nothing, when odd, we ſhall, in this Caſe, have $d^n+\overline{2d}^n+\overline{3d}|^n+\overline{4d}|^n\ \ldots\ldots\ x^n$ barely equal to $\frac{x^{n+1}}{\overline{n+1}\times d}+\frac{x^n}{2}+\frac{ndx^{n-1}}{3.4}-\frac{n\times\overline{n-1}\times\overline{n-2}\,d^3x^{n-3}}{2.3.4.5.6}$, &c. the firſt Series continued 'till it terminates, provided that the laſt Term, when $n$ is an odd Number, be rejected.

## COROL. II.

WHEREFORE, by taking $d$ equal to 1, and $n$ equal to 2, 3, 4, 5, &c. ſucceſſively, we have

$$1+2+3+4+5\ \ldots+x=\frac{x^2}{2}+\frac{x}{2}$$
$$1^2+2^2+3^2+4^2+5^2\ \ldots+x^2=\frac{x^3}{3}+\frac{x^2}{2}+\frac{x}{6}$$
$$1^3+2^3+3^3+4^3+5^3\ \ldots+x^3=\frac{x^4}{4}+\frac{x^3}{2}+\frac{x^2}{4}$$
$$1^4+2^4+3^4+4^4+5^4\ \ldots+x^4=\frac{x^5}{5}+\frac{x^4}{2}+\frac{x^3}{3}-\frac{x}{30}$$
$$1^5+2^5+3^5+4^5+5^5\ \ldots+x^5=\frac{x^6}{6}+\frac{x^5}{2}+\frac{5x^4}{12}-\frac{x^2}{12}$$
$$1^6+2^6+3^6+4\ +5^6\ \ldots+x^6=\frac{x^7}{7}+\frac{x^6}{2}+\frac{x^5}{2}-\frac{x^3}{6}+\frac{x}{42}$$

*&c.* *&c.*

## COROL. III.

MOREOVER, if $d$ be taken equal to 1, and $m$ equal to 1, our general Equation will become $2^n+3^n+4^n\ \ldots\ +x^n=\frac{x^{n+1}}{n+1}+\frac{x^n}{2}+\frac{nx^{n-1}}{3.4}$, *&c.* $-\frac{1}{n+1}-\frac{1}{2}-\frac{n}{3.4}+\frac{n\times\overline{n-1}\times\overline{n-2}}{2.3.4.5.6}$, *&c.* each Side of which being increased by Unity, and the whole multiplied by $d^n$ gives $d^n+\overline{2d}|^n+\overline{3d}|^n+\overline{4d}|^n\ \ldots\ +\overline{dx}|^n=d^n$ into $\frac{x^{n+1}}{n+1}+\frac{x^n}{2}+\frac{nx^{n-1}}{3.4}-\frac{n\times\overline{n-1}\times\overline{n-2}\,x^{n-3}}{2.3.4.5.6}$, *&c.* $-\frac{1}{n+1}+\frac{1}{2}-\frac{n}{3.4}+\frac{n\times\overline{n-1}\times\overline{n-2}}{2.3.4.5\ 6}$, *&c.*

## EXAMPLE I.

LET it be required to find the Sum of a Series, consisting of 100 Cube Numbers, whose Roots are, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, *&c.*

Here

Here $d$, the common Difference of Roots, being equal $\frac{1}{2}$, $n = 3$, and $x = 0$, let theſe Values be ſubſtituted in the Equation in *Cor.* II. and it will become $\left(\overline{\tfrac{1}{2}}\big|^{3}\right.$ *in*, $\overline{\tfrac{100}{4}}\big|^{4} + \overline{\tfrac{100}{2}}\big|^{3} + \overline{\tfrac{100}{4}}\big|^{2} = \Big)$ 3187812.5 , the Number that was to be found.

## EXAMPLE II.

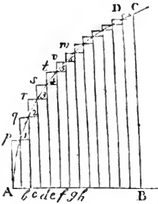
LET $n = \frac{1}{2}$, $d = \frac{1}{4}$. Then the Equation in the laſt *Corollary* will become $\overline{\tfrac{1}{4}}\big|^{\frac{1}{2}} + \overline{\tfrac{2}{4}}\big|^{\frac{1}{2}} + \overline{\tfrac{3}{4}}\big|^{\frac{1}{2}} \ldots + \overline{\tfrac{x}{4}}\big|^{\frac{1}{2}}$ $= \overline{\tfrac{1}{4}}\big|^{\frac{1}{2}} \times \overline{\frac{2 \times x^{\frac{3}{2}}}{3} + \frac{x^{\frac{1}{2}}}{2} + \frac{1}{24 \times x^{\frac{1}{2}}}}$, &c. $- \frac{339}{1920}$ very nearly; ſo that, taking $x$ equal 4, it will be $\overline{\tfrac{1}{4}}\big|^{\frac{1}{2}} + \overline{\tfrac{2}{4}}\big|^{\frac{1}{2}} + \overline{\tfrac{3}{4}}\big|^{\frac{1}{2}} + 1 = 3.0731$; which differs from the true Value by leſs than $\frac{1}{10000}$; and if more Terms had been uſed, the Anſwer would ſtill have been more exact; but never can come *accurately* true, when $n$ is negative or a Fraction, becauſe then both Series run on *ad infinitum.*

## SCHOLIUM.

THE Theorems, above found, are not only uſeful in finding the Sum of a Series of Powers, but may be of ſervice alſo in the Quadrature of Curves, &c. eſpecially as the Concluſions will be accurately true, and the Reaſoning thereupon ſcientific.

This

This I shall endeavour to shew by the following Instance; wherein A C, being supposed a Curve, whose Equation is $y = z^n$ (A B being equal $z$, and C B equal $y$) the Area A B C is required.




Let A B be divided into any Number, $x$, of equal Parts, as $Ab$, $bc$, $cd$, &c. and from the Points of Division let Perpendiculars be raised, cutting the Curve in the Points, 1, 2, 3, &c. and having made $p\,1$, $q\,2$, $r\,3$, $s\,4$, &c. parallel to A B, let the Base $Ab$, $bc$, $cd$, &c. of each of the Rectangles $pb$, $qc$, $rd$, &c. be represented by $d$: Then $b\,1$, $c\,2$, $d\,3$, &c. the Heights of those Rectangles, being Ordinates to the Curve, will be $d^n$, $\overline{2d}|^n$, $\overline{3d}|^n$, &c. respectively, each of which ∵ being multiplied by $d$, the common Base, and the Sum of all the Products taken, will give $d$ into $d^n + \overline{2d}|^n + \overline{3d}|^n \ldots \overline{xd}|^n$, (= A $p$ 1 $q$ 2 $r$, &c. C B A) for the Area of the whole circumscribing Polygon; and this Series, according to the abovesaid Theorem (*Cor.* III.) is equal to $d^{n+1}$ *in*, $\frac{nx^{n+1}}{n+1} + \frac{x^n}{2}$ &c. $= \frac{\overline{dx}|^{n+1}}{n+1} + \frac{d \times \overline{dx}|^n}{2}$, &c. or, because $dx = z$, it will be $= \frac{z^{n+1}}{n+1} + \frac{dx^n}{2}$, &c. Now, if from this the Difference of the Inscribed and circumscribed Polygons, or the

Rectangle BD $= dz^n$ be taken, there will remain $\frac{z^{n+1}}{n+1}$ $- \frac{dz^n}{2}$, for the Area of the inſcribed Polygon. Hence, it is manifeſt, that, let $d$ be what it will, the inſcribed Polygon can never be ſo great, nor the circumſcribed ſo ſmall, as $\frac{z^{n+1}}{n+1}$ $\left(= \frac{AB \times BC}{n+1}\right)$: And therefore this Expreſſion muſt be *accurately* equal to the required Curvilinear Area ACB.

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## Of Angular Sections, and ſome remarkable Properties of the Circle.

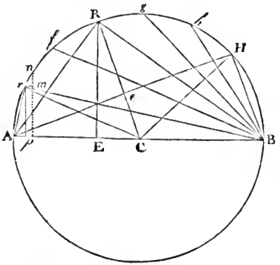
### PROPOSITION I.

*The Radius* AC, *and the Chord, Sine, or Co-ſine of an Arc, as* A$r$, *being given; to find the Chord, Sine, or Co-ſine of* AR $= m \times$ A$r$, *a Multiple of that Arc.*

LET RH be taken = AR, and the whole Arc AH be divided into as many equal Parts, A$r$, $rf$, *&c.* as there be Units in $2m$; and the Chords B$r$, B$f$, *&c.* are drawn, as alſo the Radii C$r$, CR, CH, and the Perpendiculars $rp$, RE; calling AC, 1; B$r$, $y$; C$p$, $x$; CE, X; $rp$, $u$; RE, U; A$r^2$, $z$; and $\overline{AH}^2 = Z$: Then, becauſe $AR^2$ any one of thoſe Chords, as B$f$, is to B$r$ + BR, the Sum of the 2 next it, as BC to B$r$, by a known Property of the Circle, we ſhall have $y \times Bf = Br + BR$, or $y \times Bf - Br = BR$; and for the very ſame Reaſon, $y \times BR - Bf = Bg$, and $y \times Bg - BR = Bh$, *&c. &c.* Hence, it appears, that the Values of the Chords B$f$, BR, *&c.* (which

Ee to

to a Radius equal A B, will be Co-fines of the Angles A B $f$,




A B R, &c.) may be readily had one after another, by taking continually the Product of the laft by $y$, *minus* the laft but one, for the next following: And thus are had,

$$y^2 - 2 = Bf,$$
$$y^3 - 3y = BR,$$
$$y^4 - 4y^2 + 2 = Bg,$$
$$y^5 - 5y^3 + 5y = Bh,$$
$$y^6 - 6y^4 + 9y^2 - 2 = BH,$$
&c. &c.

R. R.

And generally, fuppofing $\acute{A}y^{n-1} - \acute{B}y^{n-3} + \acute{C}y^{n-5}$, &c. to denote any one Chord of the foregoing Order, and $Ay$

$-\, B y^{n-2} + C y^{n-4}$, &c. the next to it; then the Chord next following these will be $A y^{n+1} - B y^{n-1} + C y^{n-3}$ &c. $- \acute{A} y^{n-1} + \dot{B} y^{n-3} - \acute{C} y^{n-5}$, &c. $= A y^{n+1}$ $\left.\begin{matrix} B \\ \acute{A} \end{matrix}\right\} y^{n-1} + \left.\begin{matrix} C \\ \acute{B} \end{matrix}\right\} y^{n-3} + \left.\begin{matrix} D \\ \acute{C} \end{matrix}\right\} y^{n-5}$, &c. From which (by the Method of Increments foregoing) A will come out $= 1$, $B = n$, $C = n \times \frac{n-3}{2}$, $D = n \times \frac{n-4}{2} \times \frac{n-5}{3}$, $E = n \times \frac{n-5}{3} \times \frac{n-6}{3} \times \frac{n-7}{4}$, &c. and consequently $A y^{n} - B y^{\frac{n-2}{2}} + C y^{n-4}$, &c. $= y^{n} - n y^{n-2} + n \times \frac{n-3}{2} y^{n-4} - n \times \frac{n-4}{2} \times \frac{n-5}{3} y^{n-6} + n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} y^{n-8}$, &c. wherein if $n$ be taken equal to the given Number $m$, it will become $y^{m} - m y^{m-2} + m \times \frac{m-3}{2} y^{m-4}$, &c. equal BR; but if $n$ be equal $2m$, then it will be $y^{n} - n y^{n-2} + n \times \frac{n-3}{2} y^{n-4}$, &c. equal BH; where the Series are to be continued till the Exponents become negative. Hence, because $Bf$ is equal $2x$, and the Arc $AH = m \times Af$, it follows, that the Chord HB will be $= \overline{2x}|^{m} - m \times \overline{2x}|^{m-2} + m \times \frac{m-3}{2} \times \overline{2x}|^{m-4}$, &c. and therefore, X ($=$ CE) the required Co-sine being equal $\frac{1}{2}$ HB, we have $X = \frac{\overline{2x}|^{m}}{2} - \frac{m}{2} \times \overline{2x}|^{m-2} + \frac{m}{2} \times \frac{m-3}{2} \times \overline{2x}|^{m-4} - \frac{m}{2} \times \frac{m-4}{2} \times \frac{m-5}{3} \times \overline{2x}|^{m-6} + \frac{m}{2} \times \frac{m-5}{2} \times \frac{m-6}{3} \times \frac{m-7}{4} \times \overline{2x}|^{m-8}$ &c. shewing the Relation of the Co-sines; from whence

U ( $=\sqrt{1-X^2}$) comes out $=\sqrt{1-xx}$, *in*, $\overline{2x}|^{m-1} - \frac{m-2}{1} \times \overline{2x}|^{m-3} + \frac{m-3}{1} \times \frac{m-4}{2} \times \overline{2x}|^{m-5} - \frac{m-4}{1} \times \frac{m-5}{2} \times \frac{m-6}{3} \times \overline{2x}|^{m-7}$, *&c.* Furthermore, because $\frac{BH}{2}$ is equal CE, $= \frac{y^n - ny^{n-2} + n \times \frac{n-3}{2} y^{n-4}, \text{ \&c.}}{2}$, A E will be equal to $\frac{-y^n + ny^{n-2} - n \times \frac{n-3}{2} y^{n-4}, \text{ \&c.}}{2} + 1$; and therefore $AE \times AB$ equal to $-y^n + ny^{n-2} - n \times \frac{n-3}{2} y^{n-4}$ *&c.* $+ 2 = Z$, where, if instead of $yy$, its Equal $-z+4$ ($AB^2 - Ar^2$) be substituted, it will become $Z = \pm z^{\frac{n}{2}} \mp nz^{\frac{n-2}{2}} \pm n \times \frac{n-3}{2} z^{\frac{n-4}{2}}$, *&c.* equal $\pm z^m \mp 2mz^{m-1} \pm 2m \times \frac{2m-3}{2} \times z^{m-2} \mp 2m \times \frac{2m-4}{3} \times \frac{2m-5}{3} \times z^{m-3} \pm 2m \times \frac{2m-5}{2} \times \frac{2m-6}{3} \times \frac{2m-7}{4} \times z^{m-4}$, *&c.* continued to as many Terms as there are Units in $m$. *Q. E. I.*

Otherwise,

Let the Lines $rp$, RE, be considered in a flowing State, and ($mn$) as equal to $\dot{x}$; then we shall have $\sqrt{1-xx}$ ($pr$) : 1 ($Cr$) :: $\dot{x}$ : $\frac{\dot{x}}{\sqrt{1-xx}}$ equal $rn$; and this being the Fluxion of the Arc $Ar$, that of AR (equal $m \times Ar$) will be $\frac{m\dot{x}}{\sqrt{1-xx}}$; which, for the very same Reason

that

that $\frac{\dot{x}}{\sqrt{1-xx}}$ is the Fluxion of the A $r$, muſt be equal to $\frac{\dot{X}}{\sqrt{1-X^2}}$: Whence, equally multiplying the two Denominators by $\sqrt{-1}$, we get $\frac{m\dot{x}}{\sqrt{xx-1}} = \frac{\dot{X}}{\sqrt{X^2-1}}$; where, taking the Fluent on each Side, there comes out, either, Log. $X + \sqrt{X^2-1} = m \times$ Log. $x + \sqrt{xx-1}$, or, Log. $X - \sqrt{XX-1} = m$ Log. $x - \sqrt{xx-1}$; wherefore, $X + \sqrt{X^2-1}$ and $\overline{x+\sqrt{xx-1}}|^m$, as alſo, $X - \sqrt{X^2-1}$ and $\overline{x-\sqrt{xx-1}}|^m$, the Numbers correſponding to thoſe Logarithms muſt be equal: Hence, by adding together the two Equations, we have $2X = \overline{x+\sqrt{xx-1}}|^m + \overline{x-\sqrt{xx-1}}|^m$, and by taking their Difference, $2\sqrt{X^2-1} = \overline{x+\sqrt{xx-1}}|^m - \overline{x-\sqrt{xx-1}}|^m$; from whence, by expanding the latter Part of each of the Equations into Series, and dividing the whole by 2, there will come out $X = x^m + m \times \frac{m-1}{2} x^{m-2} \times \overline{xx-1} + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} x^{m-4} \times \overline{xx-1}|^2$, &c. and $\sqrt{X^2-1} = \sqrt{xx-1}$ *in*, $m x^{m-1} + m \times \frac{m-1}{2} \times \frac{m-2}{3} x^{m-3} \times \overline{xx-1}$, &c. the former of which being reduced into ſimple Terms, gives $X = \frac{\overline{2x}|^m}{2} - \frac{m}{2} \times \overline{2x}|^{m-2} + \frac{m}{2} \times \frac{m-3}{2} \times \overline{2x}|^{m-4}$, &c. the very ſame as above found. And the latter, by multiplying by $\sqrt{-1}$, to

 take

take away the imaginary Quantities, and ſubſtituting U and $u$ inſtead of their Equals $\sqrt{1-X^2}$, $\sqrt{1-xx}$, becomes $U = u$ in, $m \times \overline{1-uu}|^{\frac{m-1}{2}} + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times -u^2 \times \overline{1-uu}|^{\frac{m-3}{2}} + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \overline{1-uu}|^{\frac{m-5}{2}} \times u^4$, &c. which, in like manner, being reduced into ſimple Terms, will be $U = mu - m \times \frac{m^2-1}{2.3} \times u^3 + m \times \frac{m^2-1}{2.3} \times \frac{m^2-9}{4.5} \times u^5 - m \times \frac{m^2-1}{2.3} \times \frac{m^2-9}{4.5} \times \frac{m^2-25}{6.7} \times u^7$, &c. *Q. E. I.*

## COROL. I.

BECAUSE the laſt Equation, as appears from the Proceſs, will hold as well when $m$ is a Fraction as when a whole Number; let $m$, or the Multiple Arch A R $(= m \times A\, r)$ be ſuppoſed indefinitely ſmall; then will $mu - m \times \frac{m^2-1}{2.3} \times u^3 + m \times \frac{m^2-1}{2.3} \times \frac{m^2-9}{4.5} \times u^5$, &c. the Sine of that Arch, or the Arch it ſelf (which in this Caſe may be conſidered as equal to it) become $mu + \frac{mu^3}{2.3} + \frac{9mu^5}{2.3.4.5} + \frac{9 \times 25 u^7}{2.3.4.5.6.7.}$, &c. and therefore the Arch A $r$ $\left(=\frac{AR}{m}\right)$ whoſe Sine is $u$, will, it is manifeſt, be $= u + \frac{u^3}{2.3} + \frac{3.3u^5}{2.3.4.5} + \frac{3.3.5.5u^7}{2.3.4.5.6.7} + \frac{3.3.5.5.7.7u^9}{2.3.4.5.6.7.8.9}$, &c.

COROL.

## COROL. II.

If $Ar$ be supposed indefinitely small, and $m$ indefinitely great, so that the Multiple Arch $m \times Ar$ $(= A)$ may be a given Quantity; then since $u$ may be considered as equal to $Ar$, $mu$ will be equal to A, and $mu - m \times \frac{m^2-1}{2.3} \times u^3$, &c. the Sine of A, equal to $mu - \frac{m^3u^3}{2.3} + \frac{m^5u^5}{2.3.4.5}$ or $A - \frac{A^3}{2.3} + \frac{A^5}{2.3.4.5} - \frac{A^7}{2.3.4.5.6.7}$, &c. because 1, 9, 25, &c. in the Factors $m^2-1$, $m^2-9$, &c. may here be rejected as indefinitely small in comparison of $m^2$.

## SCHOLIUM.

Because $\overline{x+\sqrt{xx-1}}|^m + \overline{x-\sqrt{xx-1}}|^m$ is found above to be universally $= \overline{2x}|^m - m \times \overline{2x}|^{m-2} + m \times \frac{m-3}{2} \times \overline{2x}|^{m-4} - m \times \frac{m-4}{2} \times \frac{m-5}{3} \times \overline{2x}|^{m-6}$, &c. it is evident, by Inspection, that $\overline{x+\sqrt{xx+1}}|^m + \overline{x-\sqrt{xx+1}}|^m$ will be $= \overline{2x}|^m + m \times \overline{2x}|^{m-2} + m \times \frac{m-3}{2} \times \overline{2x}|^{m-4}$, &c. and $\therefore$ $\overline{\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}}|^m + \overline{\frac{y}{2}-\sqrt{\frac{yy}{4}+rr}}|^m = y^m + my^{m-2}r^2 + m \times \frac{m-3}{2}y^{m-4}r^4$, &c. (by substituting $\frac{y}{2}$ in the room of $x$, and $rr$ in that of Unity) let $r$ and $y$ be what they will: Therefore, if $y^m + my^{m-2}r^2 +$

m

$m\times\frac{m-3}{2}\,y^{m-4}r^{4}+m\times\frac{m-4}{2}\times\frac{m-5}{3}\,y^{m-6}\,r^{6}+m\times$ $\frac{m-5}{2}\times\frac{m-6}{3}\times\frac{m-7}{4}\,y^{m-8}r^{8}$, &c. be fuppofed equal to fome given Quantity $c$, there will be given $\overline{\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}}\Big|^{m}$ $+\ \overline{\frac{y}{2}-\sqrt{\frac{yy}{4}+rr}}\Big|^{m}$, alfo $=c$; and therefore $\overline{\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}}\Big|^{2m}-2r^{2m}+\overline{\frac{y}{2}-\sqrt{\frac{yy}{4}+rr}}\Big|^{2m}$ $=cc$; wherefore, the double Rectangle of $\overline{\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}}\Big|^{m}$ into $\overline{\frac{y}{2}-\sqrt{\frac{yy}{4}+rr}}\Big|^{m}$ being $-2r^{2m}$, the Square of $\overline{\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}}\Big|^{m}-\overline{\frac{y}{2}-\sqrt{\frac{yy}{4}+rr}}\Big|^{m}$ will be $=$ $cc+4r^{2m}$, and confequently $\overline{\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}}\Big|^{m}$ $-\ \overline{\frac{y}{2}-\sqrt{\frac{yy}{4}+rr}}\Big|^{m}=\sqrt{cc+4r^{2m}}$; which Equation added to the firft gives, $2\times\overline{\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}}\Big|^{m}=$ $c+\sqrt{cc+4r^{2m}}$; and fubtracted therefrom, $2\times$ $\overline{y-\sqrt{\frac{yy}{4}+rr}}\Big|^{m}=c-\sqrt{cc+4r^{2m}}$; whence we have $\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}=\overline{\frac{c}{2}+\sqrt{\frac{cc}{4}+r^{2m}}}\Big|^{\frac{1}{m}}$, and $\frac{y}{2}-\sqrt{\frac{cc}{4}+r^{2m}}=\overline{\frac{c}{2}-\sqrt{\frac{cc}{4}+r^{2m}}}\Big|^{\frac{1}{m}}$ and therefore $y=\overline{\frac{c}{2}+\sqrt{\frac{cc}{4}+r^{2m}}}\Big|^{\frac{1}{m}}+\overline{\frac{c}{2}-\sqrt{\frac{cc}{4}+r^{2m}}}\Big|^{\frac{1}{m}}$:

Which

Which may be useful and serve as a Theorem for the Solution of certain Kind of adfected Equations, comprehended in this Form, *viz.* $y^{m} + m y^{m-2} r^{2} + m \times \frac{m-3}{2} y^{m-4} r^{4}$, &c. $= c$: For an Instance hereof, let the cubic Equation $x^{3} + b x = b$ be proposed; then, by comparing this with $y^{m} - m y^{m-2} r^{2}$, &c. we have $m = 3$, $y = x$, $m r^{2} = b$, or $r r = \frac{b}{3}$, $c = b$, and consequently $x = \overline{\frac{b}{2} + \sqrt{\frac{bb}{4} + \frac{b^{3}}{27}}}\Big|^{\frac{1}{3}} + \overline{\frac{b}{2} - \sqrt{\frac{bb}{4} + \frac{b^{3}}{27}}}\Big|^{\frac{1}{3}}$

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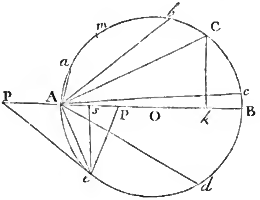
## PROPOSITION II.

*If on the Diameter* A B, *from any Point* C, *in the Circle* A C B, *whose Centre is* O, *the Perpendicular* C $k$ *be let fall, and the Arc* A C *be divided into any Number*, $m$, *of equal Parts, as* A $a$, $a$ $m$, &c. *and if the whole Periphery be also divided into the same Number of equal Parts, beginning at the Point* $a$, *as* $a$ $b$, $b$ $c$, $c$ $d$, &c. *and from any Point* P, *in the Diameter* A B, *or* A B *produced, Lines be drawn to the Points* $a$, $b$, $c$, &c. *I say*, $Pa^{2} \times Pb^{2} \times Pc^{2} \times Pd^{2}$, &c. *the continual Product of the Squares of all those Lines will be equal to* $AO^{2m} \mp AO^{m-1} \times 2Ok \times OP^{m} + PO^{2m}$.

PUT AO $=$ to 1, PO $=$ to $x$, $AP^{2} =$ to $\overline{1 \backsim x}|^{2} = v$, O$k$ $=$ to $b$, $2m =$ to $n$, and the Square of any one of the Chords A$a$, A$b$, A$c$, A$d$, &c. equal to $z$: Then, since any one of the corresponding Arcs A$a$, A$b$, A$bc$, &c. reckoned forward a certain Number of Times, brings us to the same Point C, or, is equal to AC, or AC *plus* a certain Num-

ber of Times the whole Periphery, it appears from the laſt




*Propoſition* that $\pm z^{m} \mp n z^{m-1} \pm n \times \frac{n-3}{2} z^{m-2} \mp n \times \frac{n-4}{2} \times \frac{n-5}{3} z^{m-3}$, *&c.* continued to $m$ Terms, is $= AC^2$, or becauſe $AC^2$ is $= 2 + 2b$ $(AB \times Ak)$ it will be $z^{m} - n z^{m-1} + n \times \frac{n-3}{2} z^{m-2} - n \times \frac{n-4}{2} \times \frac{n-5}{3} z^{m-3} \ldots\ldots \mp \overline{2 + 2b} = 0$, let $z$ ſtand for the Square of which of thoſe Chords you will: Wherefore, the Roots of this Equation being the Squares of the Chords $Aa$, $Ab$, $Ac$, *&c.* they muſt be all poſitive, their Sum $= n$, the Sum of their Products $n \times \frac{n-3}{2}$, of their Solids $n \times \frac{n-4}{2} \times \frac{n-5}{3}$, *&c.* by common Algebra. Now, if $se$ be made perpendicular to $AB$, we ſhall have $AP^2 + Ae^2 \pm AP \times 2As = Pe^2 = AP^2 + Ae^2 \pm AP \times \frac{Ae^2}{AO} = AP^2 + \frac{PO}{AO} \times \overline{Ae}|^2$, which

in

in Species, is $Pe^2 = v + x \times \overline{Ae}|^2$: And, for the very ſame Reaſons, $Pb^2 = v + x \times Ab^2$, $Pc^2 = v + x \times Ac^2$, &c. therefore the continual Product of $v + x \times Aa^2$ into $v + x \times Ab^2$ into $v + x \times Ac^2$, &c. is equal to $Pa^2 \times Pb^2 \times Pc^2$, &c. But in the former of theſe Products, it is evident, that when the ſeveral Factors are actually drawn into one another, the Co-efficient of the firſt Term or higheſt Power of $v$, will be 1; of the next inferior Power, the Sum of all the aboveſaid Roots $Aa^2$, $Ab^2$, &c. into $x$, of the next following, the Sum of all their Products into $x^2$, &c. and, therefore, the Sum of thoſe Roots being already found $= n$, their Products $= n \times \frac{n-3}{2}$, &c. we have $v^m + nxv^{m-1} + n \times \frac{n-3}{2} x^2 v^{m-2} + n \times \frac{n-4}{2} \times \frac{n-5}{3} x^3 v^{m-3} + n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} x^4 v^{m-4} \ldots\ldots + \overline{2+2b} \times x^m = Pa^2 \times Pb^2 \times Pc^2$, &c. Or, by ſubſtituting for $v$, its Equal $\overline{1 \backsim x}|^2$ it will be $\overline{1 \backsim x}|^n + nx \times \overline{1 \backsim x}|^{n-2} + n \times \frac{n-3}{2} x^2 \times \overline{1 \backsim x}|^{n-4} \ldots\ldots + \overline{2+2b} \times x^m = Pa^2 \times Pb^2 \times Pc^2$, &c. (becauſe $2m = n$): This in ſimple Terms is

$$\left.\begin{array}{l}
1 - nx + n \times \frac{n-1}{2} x^2 - n \times \frac{n-1}{2} \times \frac{n-2}{3} x^3, \text{ \&c.} \\
* + nx - n \times \frac{n-2}{1} \times x^2 + n \times \frac{n-2}{1} \times \frac{n-3}{2} x^3, \text{ \&c.} \\
* \quad * + n \times \frac{n-3}{2} \times x^2 - n \times \frac{n-3}{2} \times \frac{n-4}{1} x^3, \text{ \&c.} \\
* \quad * \quad * \quad + n \times \frac{n-4}{2} \times \frac{n-5}{3} x^3, \text{ \&c.} \\
\text{\&c.} \\
+ \overline{2+b} \times x^m
\end{array}\right\} = Pa^2 \times Pb^2, \text{ \&c.}$$

Which

Which contracted, by adding together the homologous Terms, becomes $1 * * *$, &c. Hence it appears, that the Coefficients do every where destroy one another, except in the first, last, and the middlemost of the said Terms; and that the middle Term would likewise vanish, if instead of $\overline{2+2b} \times x^{m}$, the corresponding Term of the above Series $\overline{1 \backsim x}|^{n} + n x \times \overline{1 \backsim x}|^{n-2}$, or that where the Exponent of $x$ is $m$, was to be added; wherefore this Term being $n \times \frac{\frac{1}{2}n-1}{2} \times \frac{\frac{1}{2}n-2}{3} \ldots \frac{1}{\frac{1}{2}n}$ into $x^{m} (=2x^{m})$ as is easy to perceive from the Law of Continuation, we have $1 + 2bx^{m} + x^{2m} = Pa^{2} \times Pb^{2} \times Pc^{2}$, &c. or, $AO^{2m} + 2Ok \times AO^{m-1} \times PO^{m} + PO^{2m} = Pa^{2} \times Pb^{2}$, &c. And, when the Point $k$ is taken on the other Side of O, $Ok$ becoming $-Ok$, $AO^{2m} - 2Ok \times AO^{m-1} \times PO^{m} + PO^{2m}$ will be equal to $Pa^{2} \times Pb^{2} \times Pc^{2}$, &c.

*Q. E. D.*

## COROL. I.

IF C be taken at B; then will $Ok = AO$, and $Pa^{2} \times Pb^{2} \times Pc^{2}$, &c. $= AO^{2m} + 2AO^{m} \times PO^{m} + PO^{2m}$; where, by taking the Square Root on each Side, we have $Pa \times Pb \times Pc$, &c. $= AO^{m} + PO^{m}$.

## COROL. II.

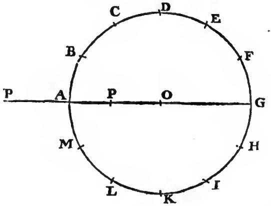
BUT if C comes into A; then A being $=0$, and $Ok = AO$, $AO^{2m} - 2Ok \times AO^{m-1} \times PO^{2m} = Pa^{2} \times Pb^{2}$, &c. will therefore become $AO^{2m} - 2AO^{m} \times$

PO$^{m}$

$PO^m + PO^{2m} = Pa^2 \times Pb^2 \times Pc^2$, &c. And $Pa \times Pb$, &c. $= AO^m \backsim PO^m$.

## COROL. III.

HENCE it is manifest, that if any Circle ABCD, &c. be divided into as many equal Parts as there are Units in $2m$ ($m$ being any whole Number what-




soever) and if in the Radius O A, produced thro' A, any one of the Points of Division, a Point as P be assumed any where, either within or without the Circle, $PA \times PC \times PE \times PG$, &c. will be $= AO^m \backsim PO^m$, $PB \times PD \times PF \times PH$, &c. $= AO^m + PO^m$, and $PA \times PB \times PC \times PD \times PE$, &c. $= AO^{2m} \backsim PO^{2m}$.

## Of the Reduction of Compound Fractions into more ſimple ones.

### PROPOSITION.

*To divide a Compound Fraction, as* $\dfrac{vx^{m-1}}{a+bx+cx^2+dx^3 \ldots x^p}$, *into as many ſimple ones as there are Units in* $p$; *ſuppoſing* $m$ *to be any whole poſitive Number, not exceeding* $p$, *and the Denominator reducible into binomial Factors.*

LET $r-x$ be any one of the given Factors into which the Denominator may be reduced, and let $\dfrac{s}{r-x}$ $+ \dfrac{A+Bx+Cx^2+Dx^3 \ldots + tx^{p-2}}{f+gx+hx^2+ix^3 \ldots + Px^{p-1}}$ be aſſumed equal to $\dfrac{vx^{m-1}}{a+bx+cx^2+dx^3 \ldots Qx^{p-1}+x^p}$; then, by Reduction, we have

$$\left.\begin{array}{l} rA+rBx+rCx^2+rDx^3+rEx^4 \ldots + rtx^{p-2} + * \\ * - Ax - Bx^2 - Cx^3 - Dx^4 \ldots\ldots\ldots\ldots - tx^{p-1} \\ sf+sgx+shx^2+six^3+skx^4 \ldots\ldots\ldots\ldots + sPx^{p-1} \\ \qquad\qquad\qquad\qquad\qquad - vx^{m-1} \end{array}\right\} = 0.$$

becauſe $r-x \cdot \overline{f+gx+hx^2}$, &c. $= a+bx+cx^2$, &c. Hence, by comparing the homologous Terms, we get $A = \dfrac{-fs}{r}$, $B = \dfrac{-fs}{r^2} - \dfrac{gs}{r}$, $C = \dfrac{-fs}{r^3} - \dfrac{gs}{r^2} - \dfrac{hs}{r}$, &c. laſtly $t$, or $sP = \dfrac{-fs}{r^{p-1}} - \dfrac{gs}{r^{p-2}} - \dfrac{hs}{r^{p-3}}$, &c. $+ \dfrac{vr^{m-1}}{r^{p-1}}$; wherefore $fs + grs + hsr^2 + isr^3 + ksr^4 \ldots + sPr^{p-1} = vr^{m-1}$ and $s = \dfrac{vr^m}{fr+gr^2+hr^3+ir^4+kr^5 \ldots + Pr^p}$:

But,

But, becaufe $\overline{r-x}\times\overline{f+gx+hx^2+ix^3\ldots\ldots+Px^{p-1}}$ is $=a+bx+cx^2+dx^3\ldots\ldots Qx^{p-1}+x^p$ we have

$$\left.\begin{array}{l} rf+rgx+rhx^2+rix^3+rkx^4\ldots\ldots rPx^{p-1} \quad * \\ * \; -fx-gx^2-hx^3-ix^4\ldots\ldots\ldots\ldots-Px^p \\ -a-bx-cx^2-dx^3-ex^4\ldots\ldots Qx^{p-1}-x^p \end{array}\right\}=0.$$

and therefore $f=\frac{a}{r}$, $g=\frac{a}{r^2}+\frac{b}{r}$, $h=\frac{a}{r^3}+\frac{b}{r^2}+\frac{c}{r}$, $i=\frac{a}{r^4}+\frac{b}{r^3}+\frac{c}{r^2}+\frac{d}{r}$, and $P=\frac{a}{r^p}+\frac{b}{r^{p-1}}+\frac{c}{r^{p-2}}$ *&c.* Whence $fr=a$, $gr^2=a+br$, $hr^3=a+br+cr^2$, $ir^4=a+br+cr^2+dr^3$, *&c.* and $\therefore Pr^p=a+br+cr^2+dr^3+er^4\ldots\ldots Qr^{p-1}$; wherefore (by adding all thefe Equations together) there will be $fr+gr^2+hr^3\ldots\ldots+Pr^p=pa+\overline{p-1}\times br+\overline{p-2}\times cr^2+\overline{p-3}\times dr^4\ldots\ldots Qr^{p-1}$; which laft Value being fubftituted in that of $s$, gives $s=\frac{vr^m}{pa+\overline{p-1}\times br+\overline{p-2}\times cr^2+\overline{p-3}\times dr^3\ldots\ldots Qr^{p-1}}$ for the Numerator of one of the required fimple Fractions; whereof the Denominator is $r-x$; from whence, by Infpection, the Numerators anfwering to the other given Factors or Denominators are eafily obtained: For, if $R-x$, $S-x$, $T-x$, *&c.* be the faid Factors into which $a+bx+cx^2+dx^3\ldots\ldots\ldots Qx^{p-1}+x^p$ is reducible, or $\overline{R-x}\times\overline{S-x}\times\overline{T-x}$, *&c.* be $=a+bx+cx^2$, *&c.* And $\frac{1}{pa+\overline{p-1}\times bR+\overline{p-2}\times cR^2+\overline{p-3}\times dR^3\ldots QR^{p-1}}$ be put $=A$, $\frac{1}{pa+\overline{p-1}\times bS+\overline{p-2}\times cS^2+\overline{p-3}\times dS^3\ldots\ldots QS^{p-1}}=B$,

*&c.*

&c. &c. it is evident, that $\frac{vx^{m-1}}{a+bx+cx^2+dx^3\ldots\ldots x^p}$ will be equal to $\frac{AvR^m}{R-x}+\frac{BvS^m}{S-x}+\frac{CvT^m}{T-x}$, &c. *Q. E. I.*

## EXAMPLE I.

LET the Fraction $\frac{x^{2-1}}{2-3x+x^2}$, $=\frac{x}{\overline{1-x}\times\overline{2-x}}$, be propofed. Then will $a=2$, $b=-3$, $c=1$, $d=0$, $e=0$, &c. $v=1$, $p=2$, $m=2$, $R=1$, $S=2$, $T=0$, &c. $A=1$, $B=-\frac{1}{2}$, and therefore $\frac{x}{2-3x+x^2}=\frac{1}{1-x}-\frac{2}{2-x}$.

## EXAMPLE II.

IF the given Fraction be $\frac{vx^{m-1}}{1\pm 2\acute{b}x^n+x^{2n}}$; then, by comparing $a+bx+cx^2+dx^3\ldots\ldots Qx^{p-1}+x^p$ with $1\pm 2\acute{b}x^n+x^{2n}$, we have $a=1$, $p=2n$, the Coefficient of the middle Term $=\pm 2\acute{b}$, and all the reft except the laft $=0$; wherefore A $\left(=\frac{1}{pa+\overline{p-1}\times br+\overline{p-2}\times cr^2\ldots\ldots Qr^{p-1}}\right)$ will in this Cafe become $=\frac{1}{2n\times 1\pm\acute{b}r^n}$, $B=\frac{1}{2n\times 1\pm\acute{b}s^n}$, &c.

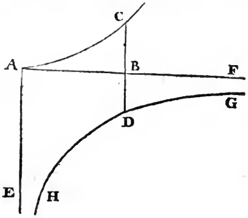
A

## A General Quadrature of Hyperbolical Curves.

### PROPOSITION

*There are two Curves* A C, H D G, *having the same common Abscissa* A B $(x)$ *whose Ordinates* B C *and* B D *are* $\frac{x^{n+r-1}}{1 \pm dx^n + x^{2n}}$, and $\frac{x^{n-r-1}}{1 \pm dx^n + x^{2n}}$; *To find the Area of each; supposing $r$ and $n$ to be any whole positive Numbers, and the Denominator $1 \pm dx^n + x^{2n}$ not reducible into two binomial Factors.*

LET $n$ be taken in $r$ as often as possible, and the Remainder be denoted by $m$; and let $Ax^{r-n} + Bx^{r-2n} +$




$Cx^{r-3n} \ldots kx^{2n+m} + lx^{n+m} + sx^m + \frac{tx^{n+m} + vx^m}{x^{2n} - dx^n + 1}$ be assumed $= \frac{cx^{n+r}}{x^{2n} - dx^n + 1}$; $c$ being any given Quantity:

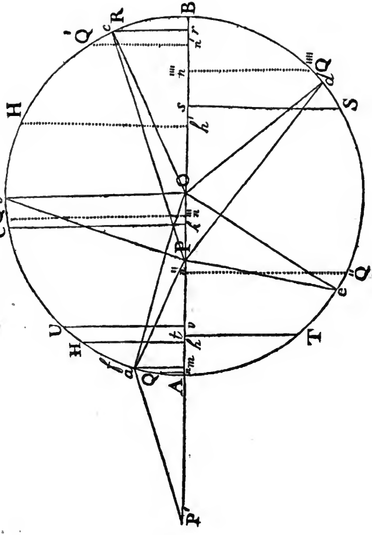


Then

Then, by reducing this Equation into one Denomination, we ſhall have $\left\{\begin{matrix}+A\\-c\end{matrix}\right\} x^{r+n} + \left\{\begin{matrix}+B\\-dA\end{matrix}\right\} x^{r} + \left\{\begin{matrix}+C\\-dB\\+A\end{matrix}\right\} x^{r-n} + \left\{\begin{matrix}+D\\-dC\\+B\end{matrix}\right\} x^{r-2n} \ldots\ldots + \left\{\begin{matrix}+s\\-dl\\+k\end{matrix}\right\} x^{2n+m} + \left\{\begin{matrix}*\\-ds\\+l\\+t\end{matrix}\right\} x^{n+m} + \left\{\begin{matrix}*\\*\\s\\v\end{matrix}\right\} x^{m} = 0$; and therefore $A = c$, $B = dA$, $C = dB - A$, $D = dC - B$, $E = dD - C$ ........ $t = ds - l$, $v = -s$.

But, now in order to conſtruct theſe ſeveral Co-efficients; with the Radius 1, and Centre O, *Fig.* 2. let the Circle AB be deſcribed; take $Ok = \frac{1}{2}d$, $Ck$ perpendicular to AB, meeting the Circle in C, and the Arch CBU to the Arch AC, as $r-m$ to $n$; and let the ſame be divided into as many equal Parts as there are Units in $\frac{r-m}{n}$, at the Points R, S, T, &c. and let $c$ be now ſuppoſed $= Ck$; then will $Ck$, $Rr$, &c. the Perpendiculars falling from thoſe Points on the Diameter AB, be equal to the ſaid required Co-efficients A, B, C, &c. reſpectively: For, ſince the Arcs AC, CR, &c. are equal, by a well known Property of the Circle $Rr$ (B) is $= \frac{2Ok}{AO} \times Ck$ ($= dA$), $-Ss$ (C) $= \frac{2Ok \times Rr}{AO} - Ck$ ($= dB - A$), $-Tt$ (D) $= \frac{2Ok \times -Ss}{AO} - Rr$ ($= dC - B$) &c. Hence,

we


A
B
O
P
R
S
T
U
H
H
Q
Q
Q
Q
Q
C

we have $\frac{cx^{n+r}}{x^{2n}-dx^n+1} = Ck \times x^{r-n} + Rr \times x^{r-2n} - Ss \times x^{r-3n} \ldots\ldots\ldots - Tt \times x^m + \frac{Uv \times x^{n+m} - Tt \times -x^m}{1-dx^n+x^{2n}}$

and therefore $\frac{x^{n+r-1}\dot{x}}{1-dx^n+x^{n2}} = \frac{\dot{x}}{cx}$ into $Ck \times x^{r-n} + Rr \times x^{r-2n} - Ss \times x^{r-3n}$, &c. $+ \frac{\frac{\dot{x}}{cx} \times \overline{Uv \times x^{n+m} - Tt \times -x^m}}{1-dx^n+x^{2n}}$;

whoſe Fluent, or $\frac{Ck \times x^{r-n}}{r-n \times c} + \frac{Rr \times x^{r-2n}}{r-2n \times c} - \frac{Ss \times x^{r-3n}}{r-3n \times c} \ldots\ldots\ldots \frac{-Tt \times x^m}{mc}$, *plus* the Fluent of $\frac{\dot{x}}{cx} \times \frac{Uv \times x^{n+m} + Tt \times x^m}{1-dx^n \times x^{2n}}$, will give the Area in the firſt Caſe.

And this Method of Solution, it is manifeſt, will hold equally, when the ſecond Term of the Diviſor is poſitive, or the given Denominator $1 + dx^n + x^{2n}$, if $k$, inſtead of being taken towards A, be taken at an equal Diſtance on the contrary Side, the Center from O towards B. But now to find the Fluent of $\frac{x^{m-1}\dot{x}}{1-dx^n+x^{2n}}$, from which that above-named will be obtained, take A$a$ to AC, as 1 to $n$, and OP $=x$; and beginning at the Point $a$, let the whole Periphery be divided into as many equal Parts, $ab$, $bc$, $cd$, &c. as there Units in $n$; letting fall the Perpendicular $am$, and putting $b = \frac{1}{2}d$ ($=$ O$k$) and O$m = f$; then, becauſe $Pa^2 \times Pb^2 \times Pc^2$ is $= 1 - 2bx^n + x^{2n}$, (as is proved in the *Propoſition* preceding the laſt) and becauſe $1 - 2fx + xx$ ($= Oa^2 - 2Om \times OP + OP^2$) is $Pa^2$, we ſhall, by feigning $1 - 2fx + xx$ equal to Nothing,

thing, get $\overline{f+\sqrt{ff-1}-x}$, and $\overline{f-\sqrt{ff-1}-x}$ for two of the $(2n)$ imaginary binomial Factors into which the said Quantity $1-2bx^n+x^{2n}$, or its Equal $\overline{f+\sqrt{ff-1}-x} \times \overline{f-\sqrt{ff-1}-x} \times$ &c. is reducible: Wherefore, if $f+\sqrt{ff-1}$ be put $=p$, and $f-\sqrt{ff-1}=q$, then will $\dfrac{p^m}{2n \times \overline{1-bp^n} \times \overline{p-x}}$, and $\dfrac{q^m}{2n \times \overline{1-bq^n} \times \overline{q-x}}$, *by the last Proposition*, be two of the simple Fractions into which $\dfrac{x^{m-1}}{1-2bx^n+x^{2n}}$ may be divided; these being added together (to take away the imaginary Quantities) give

$$\frac{p^m q+q^m p-x \times \overline{p^m+q^m}-b \times \overline{p^m q^{n+1}+q^m p^{n+1}}+bx \times \overline{p^m q^n+q^m p^n}}{2n \text{ into } \overline{1-b \times \overline{p^n+q^n}+bbp^n q^n} \text{ into } \overline{p-x} \times \overline{q-x}}$$

which, because $pq=\overline{f+\sqrt{ff-1}} \times \overline{f+\sqrt{ff-1}}$ is $=1$, will be

$$\frac{p^{m-1}+q^{m-1}-x \times \overline{p^m+q^m}-b \times \overline{q^{n+1-m}+p^{n+1-m}}+bx, \text{ \&c.}}{2n \text{ into } 1+bb-b \times \overline{p^n+q^n} \text{ into } 1-2fx+xx}:$$

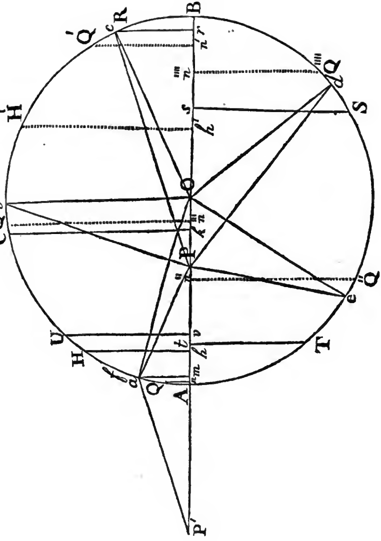
But, since $\frac{p^n+q^n}{2}$ is the Co-sine of the Multiple Arch AC $(=n \times Aa)$ and $\frac{p^m+q^m}{2}$ that of $m \times Aa$, &c. as is manifest from *Page* 109. if AH be taken $=m \times Aa$, and $Af=\overline{m-1} \times Aa$, and $am$ be put $=g$, H$b$ the Sine of AH $=$ G, and O$b=$ F; our Expression will be thus exhibited,

$$\frac{\text{Cos. of } Af-x\text{F}-b \times \text{Cos. of } Cf+bx \times \text{Cos. CH}}{ncc \times \overline{1-2fx+xx}}.$$

But as AH $-$ A$a$ is $=$ A$f$, AC $-$ AH $=$ CH, &c. we have by the Elements of *Trigonometry* the Co-sine of A$f=$ F$f+$ G$g$, Cos. of CH $=b$F $-c$G, and the Cos. of C$f=bg$G $+bf$F $+cf$G $-cg$F; and therefore our Expression will stand thus,

$\frac{Ff + Gg - b^2 fF - b^2 gG - bcfG + bcgF + \overline{b^2 F - F + bcG} \times x}{ncc \times \overline{1 - 2fx + xx}}$;
where, by fubftituting $1 - cc$ inftead of $bb$, it is, at length, reduced to $\frac{cfF + bgF + cgG - bfG + x \times \overline{bG - cF}}{cn \times \overline{1 - 2fx + xx}}$: And this drawn into $\dot{x}$, is one of the $n$ rational Fractions, (whofe Denominators are $cn \times Pa^2$, $cn \times Pb^2$, $cn \times Pc^2$, &c.) into which the Quantity $\frac{x^{m-1}\dot{x}}{1 - dx^n + x^{2n}}$, whofe Fluent we are feeking, may be divided. Now, therefore, the Fluent of $\frac{\dot{x}}{1 - 2fx + xx}$ being $\frac{1}{g}$ ($PaO$) or, $\frac{1}{g}$ into the Arch meafuring the Angle $PaO$, and that of $\frac{x\dot{x}}{1 - 2fx + xx}$, equal to the fame Arch into $\frac{f}{g}$, *plus* ($AO : Pa$) or the Hyperbolical Logarithm of $\frac{Pa}{AO}$; the Fluent of

$\frac{\overline{cfF + bgF + cgG - bfG} \times \dot{x} + \overline{bG - cF} \times x\dot{x}}{cn \times \overline{1 - 2fx + xx}}$, that of thofe Fractions whofe Denominator is $cn \times Pa^2$) will be $= \overline{\frac{Gb}{cn} - \frac{F}{n}} \times (Oa : Pa) + \overline{\frac{bF}{nc} + \frac{G}{n}} \times (PaO)$, Or, $= \frac{Hb \times Ok}{n \times Ck} - \frac{Ob}{n}$ into $(Oa : Pa)$, $+ \frac{Ob \times Ok}{n \times Ck} + \frac{Hb}{n}$ into ($PaO$): From whence the Fluents of the reft of the Fractions, which make up the required Value, whofe Denominators are $nc \times Pb^2$, $nc \times Pd^2$, &c. are determined, by Infpection, fince the Manner of Conftruction muft neceffarily be the fame in all of them. Next, from hence to find the Fluent of $\frac{x^{n+m-1}\dot{x}}{1 - 2bx^n + x^{2n}}$: For the very fame Reafon that $AH$ was taken $= m \times Aa$ in finding


A
B
O
P
P'
Q
R
S
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U
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c
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a
f
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r

ing the Fluent of $\frac{x^{m-1}\dot{x}}{1-2bx^{n}+x^{2n}}$, let $A\acute{H}$ be (now) taken $= \overline{m+n}\times A\acute{a}$, and let $\acute{H}\acute{b}$ be perpendicular to $AB$; then $\frac{\acute{H}\acute{b}\times Ok}{n\times Ck} - \frac{-O\acute{b}}{n}$ into $(Oa : Pa)$, $+ \frac{Ok\times -O\acute{b}}{n\times Ck} + \frac{\acute{H}\acute{b}}{n}$ into $(PaO)$ &c. &c. &c. will consequently be the Value sought; but $A\acute{H} - AC$ being $= AH$, $\acute{H}\acute{b}\times Ok - Ck\times -O\acute{b}$ will be $AO\times Hb$, and $-O\acute{b}\times Ok + Ck \times \acute{H}\acute{b} = Ob\times AO$, and therefore the said Value equal to $\frac{Hb}{n\times Ck}$ $(Oa : Pa) + \frac{Ob}{n\times Ck}$ $( : PaO)$ &c. &c. &c.

Now, from the two foregoing Fluents that of $\frac{\dot{x}}{cx}\times \frac{Uv\times x^{n+m} - Tt\times -x^{m}}{1-2bx^{n}+x^{2n}}$ is readily determined; and, if $TQ$ be taken $= m\times Aa$, or $ACQ = \frac{r}{n}\times AC$, and $Qn$ perpendicular to $AB$, will come out $= \frac{Qn}{nc}$ $(Oa : Pa) + \frac{On}{nc}$ $(PaO)$ &c. &c. &c. that is, if, for the same Reason that $TAQ$ is made $= m\times Aa$, $TA\acute{Q}$ be made $m\times Ab$, $TA\ddot{Q} = m\times Ac$, $TA\dddot{Q} = m\times Ad$, or $Q\acute{Q} = \acute{Q}\ddot{Q} = \ddot{Q}\dddot{Q}$, &c $= m\times ab$, and the Perpendiculars $\acute{Q}\acute{n}$, $\ddot{Q}\ddot{n}$, &c. be let fall on the Diameter $AB$, it will be

$\frac{1}{n\times Ck}$

$$\frac{1}{n\times ck}\ \text{into}\ \left\{\begin{array}{ll} Qn\,(Oa:Pa) + On\,(PaO) \\ Q'n'\,(Ob:Pb) - On'\,(PbO) \\ -Q''n''\,(Oc:Pc) + On''\,(PcO) \\ Q'''n'''\,(Od:Pd) + On'''\,(-PdO) \\ -Q''''n''''\,(Oe:Pe) - On''''\,(-PeO) \\ \quad \&c. \qquad\qquad\qquad \&c. \end{array}\right\}$$

This therefore added to $\frac{Ck\times x^{r-n}}{r-n\times c} + \frac{Rk\times x^{r-2n}}{r-2n\times c}$, &c. continued 'till the Denominators become nothing or negative, (as above found) will be the required Area in the firſt Caſe. But for the Area in the other, where the Ordinate is $\frac{x^{n-r-1}}{1\mp dx^n + x^{2n}}$, let $x$ be put equal to $\frac{1}{y}$, or $y=\frac{1}{x}$; then will $\frac{x^{r+n+1}}{1\mp dx^n + x^{2n}}$ be $= \frac{y^{+n+r-1}}{1\mp dy^n + y^{2n}}$, and $\dot{x} = \frac{-\dot{y}}{yy}$; and therefore $\frac{x^{n-r-1}\dot{x}}{1\mp dx^n + x^{2n}}$, or the Fluxion of the propoſed Area ABDHEA, equal to $\frac{-y^{n+r-1}\dot{y}}{1\mp dy^n + y^{2n}}$, and conſequently that of BFGD equal to $\frac{y^{n+r-1}\dot{y}}{1\mp dy^n + y^{2n}}$; which Expreſſion being the very ſame in Form with the Fluxion of the Area ABC; it is manifeſt, that if P, in the foregoing Conſtruction, inſtead of being taken at the Diſtance $x$ from the Centre, be put at the Diſtance $\frac{1}{x}$ $(=y)$ therefrom, as at $\acute{P}$, and the Signs of all the Indices of $x$ be changed, the Expreſſion ſhewing the ſaid Area ABC, will give that of BFGD, or the Value required in this

Case: Which therefore is $\frac{Ck}{r-n} + \frac{Rr \times x^n}{r-2n} - \frac{Ss \times x^{2n}}{r-3n}$, &c. into $\frac{x^{n-r}}{Ck}$, *plus* $\frac{1}{n \times Ck}$ into $Qn$ ($Oa : \acute{P}a$) + $On$ ($\acute{P}aO$) &c. &c. But $OP$ ($x$) being to $Oa$ ($1$), as $Oa$ to $O\acute{P}$ $\left(\frac{1}{x}\right)$, the Triangles $OPa$, and $Oa\acute{P}$, will be similar, and therefore the Angle $\acute{P}aO = OPa$, and $Oa : \acute{P}a$, $OP : Pa$, &c. wherefore the said Area will be $\frac{Ck}{r-n} + \frac{Rr \times x^n}{r-2n} - \frac{Ss \times x^{2n}}{r-3n}$, &c. into $\frac{x^{n-r}}{Ck}$

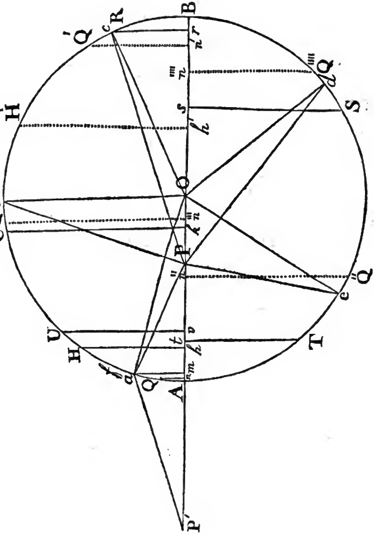
$$plus\ \frac{1}{n \times Ck}\ \text{into} \left\{ \begin{array}{l} Qn\,(OP : Pa) + On\,(\;OPa) \\ \acute{Q}\acute{n}\,(OP : Pb) - \acute{O}\acute{n}\,(\;OPb) \\ -\ddot{Q}\ddot{n}\,(OP : Pc) + O\ddot{n}\,(\;OPc) \\ \dddot{Q}\dddot{n}\,(OP : Pd) + O\dddot{n}\,(-OPd) \\ -\ddddot{Q}\ddddot{n}\,(OP : Pe) - O\ddddot{n}\,(-OPe) \\ \quad\&c. \qquad\qquad \&c. \end{array} \right\}$$

*Q. E. I.*

## COROL. I.

HENCE the Area of a Curve, whose Abscissa is $x$, and Ordinate $\frac{x^{n+r-1}}{g^{2n} \pm g^{n-1}dx^n + x^{2n}}$ may be easily obtained: For, let the Radius of the Circle A B, now be denoted by $g$, the rest as before; then, since every Term in the required Area must consist of the same Number ($r-n$) of Dimensions, by substituting the several Powers of $g$ for those of A O, or Unity in our former Area, it will become

$\frac{x^{r-n}}{Ck}$


B
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$\frac{x^{\overline{r-n}}}{Ck}$ into $\frac{Ck}{r-n} + \frac{Rr \times g^n x^{-n}}{r-2n} - \frac{Ss \times g^{2n} x^{-2n}}{r-3n}$, &c. $+ \frac{g^{\overline{r-n}}}{n \times Ck}$ into $Qn\,(Oa : Pa) + On\,(PaO) +$ &c. &c. for the Area in this Caſe.

## C O R O L. II.

HENCE, alſo, may the Area of a Curve, whoſe Abſciſſa is $z$, and Ordinate $\frac{z^{p + \frac{r}{n}p - 1}}{a^{2p} \pm a^{p-1} f z^p + z^{2p}}$, where $p$ denotes any Number at pleaſure, and $r$ and $n$ as above, be eaſily derived: For, putting $z^{\frac{p}{n}} = x$, $a^{\frac{p}{n}} = g$, and $d = f \times a^{\frac{p}{n} - 1}$, or $f = d \times g^{\frac{n}{p} - 1}$, we have $\dot{z} z^{p-1} = \frac{n}{p} \dot{x} x^{n-1}$ and $\frac{z^{p + \frac{r}{n}p - 1} \dot{z}}{a^{2p} \pm a^{p-1} f z^p + z^{2p}} = \frac{\frac{n}{p} x^{n+r-1} \dot{x}}{g^{2n} \pm g^{n-1} d x^n + x^{2n}}$ wherefore, if in the foregoing Area, anſwering to the Abſciſſa $x$, and Ordinate $\frac{x^{n+r-1}}{g^{2n} \pm d g^{n-1} x^n + x^{2n}}$, theſe Values of $d$, $g$, $x$, be reſpectively ſubſtituted, and the whole be multiplied by $\frac{n}{p}$, it will become the Area in the preſent Caſe, which therefore will be $\frac{n z^{\frac{r}{n} \times p - p}}{p \times Ck}$ into $\frac{Ck}{r-n} + \frac{Rr \times a^p z^{-p}}{r-2n} + \frac{-Ss \times a^{2p} z^{-2p}}{r-3n}$, &c. continued 'till the Denominators become nothing or negative,

+

$$+\frac{a^{\frac{r}{n}\times p-p}}{p\times Ck}\ in\left\{\begin{array}{ll} Qn\,(OA:Pa)+On\,(\ PaO) \\ \acute{Q}\acute{n}\,(OA:Pb)-O\acute{n}\,(\ PbO) \\ -\ddot{Q}\ddot{n}\,(OA:Pc)+O\ddot{n}\,(\ PcO) \\ \dddot{Q}\dddot{n}\,(OA:Pd)+O\dddot{n}\,(-PdO) \\ -\ddddot{Q}\ddddot{n}\,(OA:Pe)-O\ddddot{n}\,(-PeO) \\ \&c. \qquad\qquad \&c. \end{array}\right\}$$

Where, according to the foregoing Conſtruction, AO ſhould $=a^{\frac{p}{n}}$, $Ok=\frac{1}{2}f\times a^{\frac{p}{n}-1}$, and $PO=z^{\frac{p}{n}}$; but ſince each Term in the Area, when actually divided by the common Diviſor $Ck$, will be affected both in its Numerator and Denominator by one ſingle Dimenſion or Power of Lines exhibited in the Circle, whoſe Ratios do not at all depend on the Magnitude of that Circle, it matters not, whether AO, $Ok$, and PO be taken exactly equal to thoſe Quantities, or to others in the ſame Proportion, as $a$, $\frac{1}{2}f$, and $\overline{\frac{z}{a}}|^{\frac{p}{n}}\times a$, or 1, $\frac{\frac{1}{2}f}{a}$, and $\overline{\frac{z}{a}}|^{\frac{p}{n}}$, provided the reſt of the Conſtruction be retained. The like will hold in the Area of the Curve whoſe Abſciſſa is $z$, and Ordinate $\frac{z^{p-\frac{r}{n}p-1}}{a^{2p}\pm fa^{p-1}z^{p}+z^{2p}}$, which by proceeding in the ſame Manner, from the ſecond Caſe, will come out

$$\frac{nz^{-\frac{r}{n}p+p}}{p\times Ck\,a^{2p}}\ in\ \frac{Ck}{r-n}+\frac{Rr\times z^{p}a^{-p}}{r-2n}+\frac{-Ss\times z^{2p}a^{-2p}}{r-3n},\ \&c.$$

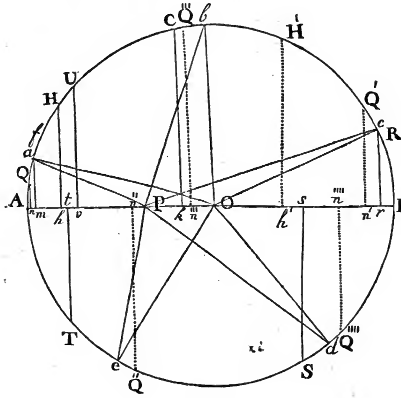
'till the Denominator becomes nothing or negative,

$$+\frac{a^{-\frac{r}{n}p-p}}{p\times Ck}\ in\left\{\begin{array}{l} Qn\,(OP:Pa)+On\,(\ aPO) \\ Q'n'\,(OP:Pb)-On'\,(\ bPO) \\ -Q''n\,(OP:Pc)+On''\,(\ cPO) \\ Q'''n'''\,(OP:Pd)+On'''\,(-dPO) \\ -Q'n''\,(OP:Pe)-On'''\,(-ePO) \\ \text{\&c.} \qquad\qquad\qquad\qquad \text{\&c.} \end{array}\right\}$$

Hence, to find the Area ABCA of a Curve, whoſe Abſciſſa is $z$, and Ordinate $\frac{z^{p+\frac{r}{n}p-1}}{a^{2p}\pm fa^{p-1}z^{p}+z^{2p}}$, or the Area BFGD of that whoſe Abſciſſa is $z$ (= A B) and Ordinate $\frac{z^{p-\frac{r}{n}p-1}}{a^{2p}\pm fa^{p-1}z^{p}+z^{2p}}$ (ſee the foregoing *Fig.*), we have this Conſtruction. From the Centre O of the Circle ACB, whoſe Radius is = 1, take the Point $k$ in the Diameter AB, towards A or B, according as the Sign of $fa^{p-1}\ z^{p}$ is − or +, ſo that O$k$ may = $\frac{f}{2a}$, and draw the Perpendicular C$k$ to the Point $k$ in that Diameter, meeting the Circle in C; make A$a$ to AC, as 1 to $n$, and ACQ to AC, as $r$ to $n$, and CR, RS, ST, &c. each equal to AC; and, beginning at the Point $a$, let the whole Circle be divided into as many equal Parts as there are Units in $n$, as $ab$, $bc$, $cd$, &c. take OP = $\overline{\tfrac{z}{a}}\Big|^{\frac{p}{n}}$, and each of the Arches $QQ'$, $Q'Q''$, &c. = the Arch $ab$ × into the Remainder of $r$ divided by $n$, and draw P$a$, O$a$, P$b$, O$b$, &c. and the Perpendiculars R$r$, S$s$, T$t$, &c. Q$n$, Q$'n'$, Q$''n''$, &c. to the Diameter AB; then

then will the Areas be refpectively as above exhibited. And it muft be obferved, that this Solution holds in all Cafes where $f$ is lefs than $2a$, and $r$ and $n$ whole pofitive Numbers.

*Note.* That (OP : P$a$) is put to denote the hyperbolical Logarithm of $\frac{Pa}{OP}$, and ($a$PO) the Meafure of the Angle $a$PO in Parts of Radius or Unity; the like is to be underftood of any other.



## SCHOLIUM.

THE above Solution being ſomewhat intricate, others by infinite Series, where they will converge, may be thought preferable; but as the greateſt, if not the only, Difficulty in what has been here delivered, lies in finding the Fluent of $\frac{x^{m-1}\dot{x}}{1-2bx^n+x^{2n}}$, or $\frac{x^{n+m-1}}{1-2bx^n+x^{2n}}$, it may be proper (before any thing is offered on this Head) to add a different Method whereby the ſaid Fluent of $\frac{x^{n+m-1}}{1-2bx^n+x^{nn}}$, may, in the Form it ſtands above, be more eaſily inveſtigated.

In order thereto, the firſt Conſtruction of the Points C, R, S, *&c.* $a$, $b$, $c$, *&c.* *&c.* being premiſed, let the Sines of the Arches AC, A$a$, A$b$, A$c$, *&c.* be called B, C, D, E, *&c.* and their Co-ſines $b$, $\frac{1}{2}c$, $\frac{1}{2}d$, $\frac{1}{2}e$, *&c.* reſpectively: Then, becauſe $1-cx+xx$ is $=\mathrm{P}a^2$, $1-dx+xx=\mathrm{P}b^2$, $1-ex+xx=\mathrm{P}c^2$, *&c.* and $\mathrm{P}a^2\times\mathrm{P}b^2\times\mathrm{P}c^2\times\mathrm{P}d^2$, *&c.* $=1-2bx^n+x^{2n}$, it follows, that the Sum of the Logarithms of $1-cx+xx$, $1-dx+xx$, *&c.* muſt be equal to the Logarithm of $1-2bx^n+x^{2n}$, and therefore $\frac{2x\dot{x}-c\dot{x}}{xx-cx+1}+\frac{2x\dot{x}-d\dot{x}}{xx-dx+1}$, *&c.* the Sum of their Fluxions $=\frac{2n\dot{x}\times\overline{x^{2n-1}-bx^{n-1}}}{1-2bx^n+x^{2n}}$, the Fluxion of that Logarithm. Hence, by taking each Side of the Equation from $\frac{2n\dot{x}}{x}$, and dividing the whole by $\dot{x}$, we have $\frac{1}{x}$ *in* $\frac{2-cx}{xx-cx+1}$

+

$+\frac{2-dx}{xx-dx+1}$, &c. $=\frac{2n\times\overline{x^{-1}-bx^{n-1}}}{1-2bx^{n}+x^{2n}}$; this Equation being multiplied by $x^n$, and the former by $\frac{b}{x}$, and the Products added together, there will be $b$ into $\frac{2x-c}{1-cx+xx}+\frac{2x-d}{1-dx+xx}$, &c. $+\ x^{n-1}$ into $\frac{2-cx}{1-cx+xx}+\frac{2-dx}{1-dx+xx}$, &c. $\left(=\frac{\overline{1-bb}\times 2nx^{n-1}}{1-2bx^{n}+x^{2n}}\right)=\frac{2nB^2x^{n-1}}{1-2bx^{n}+x^{2n}}$. Now, to reduce $x^{n-1}\times\frac{2-cx}{xx-cx+1}+\frac{2-dx}{xx-dx+1}$, &c. to lower Dimensions; Let $\acute{A}x^{n-2}+\acute{B}x^{n-3}+\acute{C}x^{n-4}\ldots\ldots\acute{k}x^2+\acute{l}x+\acute{s}+\frac{\acute{t}x+\acute{v}}{xx-cx+1}$ be assumed $=-\frac{1}{2}\times\frac{x^{n-1}\times\overline{2-cx}}{1-cx+xx}$, or $\frac{\frac{1}{2}cx^{n}-x^{n-1}}{xx-cx+1}$; then, by bringing the Equation into one Denomination, and equating the like Terms, we have $\acute{A}=\frac{1}{2}c$, $\acute{B}=c\acute{A}-1$, $\acute{C}=c\acute{B}-\acute{A}$, $\acute{D}=c\acute{C}-\acute{B}$, &c. and $v=-s$: But, these Equations, it is manifest, (from a known Property of the Circle) likewise express the Relation of the Co-sines of the Arches $Aa$, $2Aa$, $3Aa$, &c. therefore ($\acute{A}=\frac{1}{2}C$) being the Co-sine of the first of those Arches, and $\acute{B}$ that of the second, the Values of $\acute{C}$, $\acute{D}$, $\acute{E}$, &c. which entirely depend on these, must consequently be equal to the Co-sines of the rest of those Arches respectively; and therefore $\acute{t}$ equal to the Co-sine of $n\times Aa$, and $-\acute{v}=$ that of $\overline{n-1}\times Aa$. Wherefore, if $\ddot{A}x^{n-2}+$

$\overset{\prime\prime}{B} x^{n-3} + \overset{\prime\prime}{C} x^{n-4} \ldots\ldots \overset{\prime\prime}{k} x^{2} + \overset{\prime\prime}{l} x + \overset{\prime\prime}{s} + \frac{\overset{\prime\prime}{t} x + \overset{\prime\prime}{v}}{xx - dx + 1}$, be, in like Manner, aſſumed $= - \frac{1}{2} \times \frac{x^{n-1}\overline{2-dx}}{xx-dx+1}$, and $\overset{\prime\prime\prime}{A} x^{n-2} + \overset{\prime\prime\prime}{B} x^{n-3} + \overset{\prime\prime\prime}{C} x^{n-4}$, &c. $= - \frac{1}{2} \times \frac{x^{n-1} \times \overline{2-ex}}{xx-ex+1}$ &c. &c. it follows, for the very ſame Reaſon, that $\overset{\prime\prime}{A}$, $\overset{\prime\prime}{B}$, $\overset{\prime\prime}{C}$, $\overset{\prime\prime}{t}$, $-\overset{\prime\prime}{v}$, will be the Co-ſines of the Arches A $b$, 2 A $b$, 3 A $b$, $n \times$ A $b$, and $\overline{n-1} \times$ A $b$, and $\overset{\prime\prime\prime}{A}$, $\overset{\prime\prime\prime}{B}$, $\overset{\prime\prime\prime}{C}$, $\overset{\prime\prime\prime}{t}$, $-\overset{\prime\prime\prime}{v}$, thoſe of A $c$, 2 A $c$, 3 A $c$, $n \times$ A $c$, $n - 1 \times$ A $c$. &c. &c. reſpectively; and we ſhall, by adding together the ſaid Equations, have $- \frac{1}{2} \times x^{n-1} \times \overline{\frac{2-cx}{1-cx+xx} + \frac{2-dx}{1-dx+xx}}$ &c. $= x^{n-2} \times \overline{\overset{\prime}{A} + \overset{\prime\prime}{A} + \overset{\prime\prime\prime}{A}}$, &c. $+ x^{n-3} \times \overline{\overset{\prime}{B} + \overset{\prime\prime}{B} + \overset{\prime\prime\prime}{B} + \overset{\prime\prime\prime\prime}{B}}$, &c. $+ x^{n-4} \times \overset{\prime}{C} + \overset{\prime\prime}{C} + \overset{\prime\prime\prime}{C}$, &c. &c. $+ \frac{\overset{\prime}{t} x + \overset{\prime}{v}}{1-cx+xx} + \frac{\overset{\prime\prime}{t} x + \overset{\prime\prime}{v}}{1-dx+xx}$, But, $\overset{\prime}{A}$, $\overset{\prime\prime}{A}$, $\overset{\prime\prime\prime}{A}$, &c. being the Co-ſines of the Arches A $a$, A $b$, A $c$, &c. and therefore the Roots of an Equation, expreſſing the Relation of the Co-ſine of an Arch, to that of another Arch $n$ times as great, wherein the ſecond Term is always wanting (*vide* p. 106.) their Sum muſt therefore be equal to nothing, by common *Algebra*; which is evident even by Inſpection, when $n$ is an even Number; for, then every one of the Points $a$, $b$, $c$, &c. above A B having another Point, of the ſame Conſtruction, diametrically oppoſite to it, the Sines, as well as the Co-ſines, anſwering to thoſe Points, muſt be equal and contrary, and therefore

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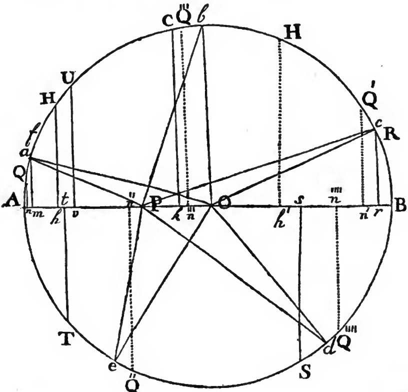
destroy each other. In like manner $\acute{B}$, $\acute{\acute{B}}$, $\acute{\acute{\acute{B}}}$, &c. being the Co-sines of $2\,Aa$, $2\,Ab$, $2\,Ac$, &c. or the Roots of such another Equation, they must also destroy one another, &c. &c. Hence our Equation is reduced to $-\frac{1}{2}\,x^{n-1} \times \overline{\frac{2-cx}{1-cx+xx} + \frac{2-dx}{1-dx+xx}}$, &c. $= \frac{\acute{\imath}x+\acute{v}}{1-cx+xx} + \frac{\acute{\acute{\imath}}x+\acute{\acute{v}}}{1-dx+xx} + \frac{\acute{\acute{\acute{\imath}}}x+\acute{\acute{\acute{v}}}}{1-ex+xx}$, &c. But since $\acute{\imath}$ is found to be the Co-sine of $n \times Aa$




or

or AC, $\overset{\prime\prime}{t}$ that of $n \times \mathrm{A}\,b$ or ABC, $\overset{\prime\prime\prime}{t}$ that of $n \times \mathrm{A}\,c$ or ABCBC, &c. $-\overset{\prime}{v} =$ that of $\overline{n-1} \times \mathrm{A}\,\dot{a}$, $-\overset{\prime\prime}{v} =$ that of $\overline{n-1} \times \mathrm{A}\,b$, &c. we have $\overset{\prime}{t} = \overset{\prime\prime}{t} = \overset{\prime\prime\prime}{t}$, &c. $= \mathrm{O}\,k = b$, and $\overset{\prime}{v} = -\mathrm{B\,C} - \frac{bc}{2}$, $\overset{\prime\prime}{v} = -\mathrm{B\,D} - \frac{bd}{2}$, &c. and therefore $-\frac{x}{2}^{n-1} \times \overline{\frac{2-cx}{1-cx+xx} + \frac{2-dx}{1-ax+xx}}$, &c. $= \frac{bx - \mathrm{BC} - \frac{1}{2}bc}{1-cx+xx} + \frac{bx - \mathrm{BD} - \frac{1}{2}bd}{1-dx+xx}$ &c. or $x^{n-1} \times \overline{\frac{2-cx}{1-cx+xx} + \frac{2-dx}{1-ax+xx} + \frac{2-ex}{1-ex+xx}}$, &c. $= \frac{bc + 2\mathrm{BC} - 2bx}{1-cx+xx} + \frac{bd + 2\mathrm{BD} - 2bx}{1-dx+xx}$, &c. which being fubftituted, inftead thereof in $x^{n-1} \times \overline{\frac{2-cx}{1-cx+xx}}$, &c. $= \frac{2n\mathrm{B}^2 x^{n-1}}{1-2bx^n+x^{2n}}$ (as abovefound) and the whole divided by 2 B, we fhall have $\frac{\mathrm{C}}{1-cx+xx} + \frac{\mathrm{D}}{1-dx+xx} + \frac{\mathrm{E}}{1-ex+xx} + \frac{\mathrm{F}}{1-fx+xx}$, &c. $= \frac{n\mathrm{B}x^{n-1}}{1-2bx^n+x^{2n}}$, and, confequently $\frac{n\mathrm{B}x^{n+m-1}}{1-2bx^n+x^{n2}} = \frac{\mathrm{C}x^m}{1-cx+xx} + \frac{\mathrm{D}x^m}{1-ax+xx} + \frac{\mathrm{E}x^m}{1-ex+xx} + \frac{\mathrm{F}x^m}{1-fx+xx}$, &c. Let now $\acute{\mathrm{A}}x^{m-2} + \acute{\mathrm{B}}x^{m-3} + \acute{\mathrm{C}}x^{m-4}$ $\vdots \ldots\ldots \acute{l}x + \acute{s} + \frac{\acute{t}x + \acute{v}}{xx-cx+1}$ be affumed $= \frac{\mathrm{C}x^m}{xx-cx+1}$; then, by Reduction, &c. we fhall get $\acute{\mathrm{A}} = \acute{\mathrm{C}}$, $\acute{\mathrm{B}} = c\,\acute{\mathrm{A}}$, $\acute{\mathrm{C}} = c\,\acute{\mathrm{B}} - \acute{\mathrm{A}}$, $\mathrm{D} = c\,\acute{\mathrm{C}} - \acute{\mathrm{B}}$, &c. &c. where it appears, from the afore-named Property of the Circle, that $\acute{\mathrm{A}}$, $\acute{\mathrm{B}}$, $\acute{\mathrm{C}}$, .... $\acute{t}$, $-\overset{\prime}{v}$, are the Sines of $\mathrm{A}\,a$, $2\,\mathrm{A}\,a$, $3\,\mathrm{A}\,a$, ... $m \times \mathrm{A}\,a$, $\overline{m-1}$

A $a$

A $a$ refpectively. Hence, if $\overset{''}{A}x^{m-2} + \overset{''}{B}x^{m-3} + \overset{''}{C}x^{m-4} \ldots\ldots \overset{''}{l}x + \overset{''}{s} + \frac{\overset{''}{t}x + \overset{''}{v}}{xx - xd + 1}$ be put $= \frac{Dx^m}{xx - dx + 1}$, it is manifeft, that $\overset{''}{A}$, $\overset{''}{B}$, $\overset{''}{C} \ldots\ldots \acute{t}$, $-\acute{v}$ will be the Sines of A $b$, 2 A $b$, 3 A $b$ .... $m \times$ A $b$ and $\overline{m-1} \times$ A $b$ refpectively, &c. &c. Therefore, as it is evident from the above Reafons, that $\acute{A} + \overset{''}{A}$, &c. and $\acute{B} + \overset{''}{B}$, &c. and $\acute{C} + \overset{''}{C}$, &c. &c. muft all be equal to Nothing, we have $\frac{nBx^{n+m-1}}{1 - 2lx^n + 2x^n} =$

$$\frac{x \times \text{Sine of } m \times Aa - \text{Sine of } \overline{m-1} \times Aa}{1 - cx + xx} + \frac{x \times \text{Sine of } m \times Ab - \text{Sine of } \overline{m-1} \times Ab}{1 - dx + xx}$$

&c. which, becaufe A H is $= m \times$ A $a$ (by Conftruction), will become $\frac{nBx^{n+m-1}}{1 - 2bx^n + x^{2n}} = \frac{Hb \times x - \text{Sine}\,(AH - Aa).}{1 - cx + xx}$, &c. $= \frac{Hb \times x + am \times Ob - Hb \times Om}{1 - cx + xx}$, &c. Wherefore, the Fluents of $\frac{x\dot{x}}{1 - cx + xx}$ and $\frac{\dot{x}}{1 - cx + xx}$ being $(Pa : Oa) + \frac{Om}{am}$ $(PaO)$ and $\frac{1}{am}$ $(PaO)$ refpectively, that of $\frac{Hb \times x\dot{x} + \overline{am \times Ob - Hb \times Om} \times \dot{x}}{nB \times 1 - cx + xx}$, &c. &c. or its Equal $\frac{x^{n+m-1}\dot{x}}{1 - 2bx^n + x^{2n}}$, muft confequently be $= \frac{Hb}{nB}$ $(Pa : Oa)$ $+ \frac{Ob}{nB}$ $(PaO)$, &c. &c. which is the very fame as was before found.

Having thus far effected what was propofed, it remains next to lay down a Method for finding the aforefaid Areas A B C, B F G D, in a more eafy Manner, by Approximation

 and

and infinite Series, when that can be done. In order to this, the foregoing Conſtruction of the Points *k*, P, and C, being ſtill retained, let CR, RS, ST, TU, *&c.* be each equal to A C, and R *r*, S *s*, T *t*, *&c.* perpendicular to AB. Then will $\frac{nz^{\frac{r}{n}p+p}}{pa^{2p}\times Ck}$ into $\frac{Ck}{r+n} + \frac{Rr\times z^{p}a^{-p}}{r+2n} + \frac{-Ss\times z^{2p}a^{-2p}}{r+3n}$ $\frac{-Tt\times z^{3p}a^{-3p}}{r+4n} + \frac{Uv\times z^{4p}a^{-4p}}{r+5n}$, *&c. ad infinitum*, or $\frac{nz^{\frac{r}{n}p-p}}{p\times Ck}$ into $\frac{Ck}{r-n} + \frac{Rr\times z^{-p}a^{p}}{r-2n} + \frac{-Ss\times z^{-2p}a^{2p}}{r-3n}$, *&c.* be $=$ A B C A, or the Area of the Curve whoſe Abſciſſa is $z$, and Ordinate $\frac{z^{p+\frac{r}{n}\times p-1}}{a^{2p}\pm fa^{p-1}z^{p}+z^{2p}}$; and $\frac{nz^{p-\frac{r}{n}p}}{pa^{2p}\times Ck}$ into $\frac{Ck}{r-n} + \frac{Rr\times z^{p}a^{-p}}{r-2n} + \frac{-Ssz^{2p}a^{-2p}}{r-3n} + \frac{-Tt\times z^{3p}a^{-3p}}{r-4n}$ *&c.* or $\frac{nz^{-p-\frac{r}{n}p}}{p\times Ck}$ *in* $\frac{Ck}{r+n} + \frac{Rr\times z^{-p}a^{p}}{r+2n} + \frac{-Ss\times z^{-2p}a^{2p}}{r+3n}$ $\frac{-Tt\times z^{-3p}a^{3p}}{r+4n}$, *&c.* $=$ B F G D, or the Area of the Curve whoſe Abſciſſa is $z$, and Ordinate $\frac{z^{p-\frac{r}{n}p-1}}{a^{2p}\pm fa^{p-1}z^{p}+z^{2p}}$. The Reaſons whereof, from the former Part of the foregoing Solution, will appear manifeſt.

*F I N I S.*

# E R R A T A.

PAGE 5. l. 17. for *q r* read *g r*, p. 28. l. 14. r. $\frac{vv}{2} = \frac{2rx^{n+1}}{\overline{n+1} \times a^{n}}$, l. 18. for $a^2v^2$ r. $x^2v^2$, l. laſt, for $\sqrt{m^2,\ \&c.}$ r. $\frac{\dot{z}}{\sqrt{m^2,\ \&c.}}$, p. 31. l. 5. for *the Ratio*, r. *which is to Unity in the Ratio*, p. 32. l. 18. for *than*, r. *than as*, p. 42. l. 24. for R*s* r. *ks*, l. 25. for *vr* r. *sr*, p. 48. l. 25. for S C r. *l*C, p. 49. l. 16. for 5 *or* 6 r. 7 *or* 8, p. 51. l. 19. for 52′ 50″ r. 51′ 52″. l. penult. for 52′ 51″ r. 51′ 52″, p. 53. l. 20. for $\frac{b}{g}$ r. $\frac{v}{g}$, p. 57. l. 24. for *Specifick Gravity* r. *Specifick Gravity in Air*, p. 61 l. 20. for $\frac{y^2}{2p}$ r. $\frac{y^2}{p}$, p. 67. l. 18. for *of* r. *to*, p. 76. l. 12. for *of n* r. *n*, p. 85. l. 10. for $1 - \frac{x}{y}$ r. $\frac{x}{y}$, p. 93. l penult. for *Allowances* r. *Allowance*, p. 101. l. laſt, for 2, 3, &c. r. 1, 2, 3, &c. p. 103. l. 2. for $x=0$ r. $x=100$, l. 3. for *Cor.* II. r. *Cor.* III. p. 104. l. 24. for $nx^{n+1}$ r. $x^{n+1}$, l. penult. for $dx^{n}$ r. $dz^{n}$, p. 105. l. 2. for $dzn$ r. $dz^{n}$, l. 15. for *be* r. *are*, and for *are* r. *be*, l. 18 for A H r. A R, p. 113. l. 24. for $\overline{1 \backsim x}|^2 v$ r. $\overline{1 \backsim x}|^2 = v$, p. 121. l. laſt, for $x^{n-r}$ r. $x^{n+r}$.

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